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Discrete time output feedback sliding mode tracking control for systems with uncertainties

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SUMMARY

This paper describes a method for designing discrete time static output feedback sliding mode tracking controllers for uncertain systems that are not necessarily minimum phase or of relative degree one. In this work, a procedure for realizing discrete time controllers via a particular set of extended outputs is presented for systems with uncertainties. The conditions for existence of a sliding manifold guaranteeing a stable sliding motion are given. A procedure to synthesize a control law that minimizes the effect of the disturbance on the sliding mode dynamics and the augmented outputs is given. The proposed control law is then applied to a benchmark aircraft problem taken from the literature that represents the lateral dynamics of a F-14 aircraft under powered approach. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

This paper will examine the design of discrete time sliding mode tracking controllers using output information for both square and nonsquare, possibly nonminimum phase systems. Compared with continuous time sliding mode control strategies, the design problem in discrete time is much less mature. Other than the early work in [1], much of the literature assumes all states are available [2–4]. Discrete sliding mode control schemes that have restricted themselves to output measurements alone have often been observer-based schemes, with or without disturbance estimation [5, 6]. Recent exceptions have been the work in [7] that considers both static and dynamic output feedback; the work in [8] that considers decentralized discrete sliding mode output feedback control for a class of interconnected systems and the discrete time versions of certain higher-order sliding-mode control schemes [9, 10].

For static output feedback-based sliding mode control in continuous time, it is an a priori requirement that the system under consideration be minimum phase and that a particular subsystem be output feedback stabilizable [11, 12]. In the case of discrete time static output feedback-based sliding mode control, the minimum phase requirement can be relaxed under certain conditions, and a controller design can be performed [13, 14]. In [14], a novel sliding surface has been introduced, and a discrete time output feedback-based sliding mode control design for systems with matched uncertainties and that are not necessarily relative degree one has been given. It has been shown that the reduced order sliding motion associated with this sliding surface is not dependent on the invariant zeros of the system that hence is not required to be minimum phase [14].

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For discrete time systems, by using the output signal at the current time instant together with a limited amount of information from previous sample instants, a method for synthesizing static output feedback discrete time sliding mode controllers has been given for systems with uncertainties in [13]. It has been shown that by using extended outputs, discrete time output feedback-based sliding mode control for nonminimum phase systems is possible. It has also been shown that, by extending the outputs, the relative degree condition associated with the solution of the existence problem can be relaxed for output feedback-based sliding mode control. By using the extended outputs, it has been shown that the sliding dynamics is a function of the disturbance and that the ideal sliding mode as defined for a nominal system is not possible [13]. In this case, the reduced order dynamics has been shown to be bounded about a region around the sliding surface. The extended outputs chosen for the control design were such that the effect of the disturbance on the sliding surface was nullified [13]. The aforementioned condition is restrictive because it might not always be possible to choose the extended outputs such that the disturbance channel spans the null space of a particular lower triangular matrix.

In this paper, the aforementioned restriction will be removed and a discrete time output feedback-based sliding mode control design for providing tracking for systems that are possibly nonminimum phase and with uncertainty will be given utilizing extended outputs for the more general case where the sliding surface is a function of the disturbance. The control design will utilize integral action to achieve tracking. The work in this paper will thus build on the work in [13]. The control law chosen will be such that the upper bound on the H_2 norm of a particular transfer function relating the disturbance input to the output is minimized. It will also be shown that the control law chosen will force the states of the uncertain system to the sliding surface in a finite time and will maintain an ideal sliding motion. The control law will then be applied to a benchmark aircraft problem taken from the literature.

The aircraft model considered is that of the lateral dynamics of an F-14 aircraft under powered approach [15, 16]. As stated in [15], the control task involves landing an aircraft on an aircraft carrier that requires precise control because the aircraft tail hook must engage one of the arrestment wires spaced 40 ft apart on the deck. The aircraft with an approach speed of 137 Kn, and with a -3 deg glide path under no ship motion conditions will clear the ship by 11 ft and will touch down 1 s later with an impact velocity of 13 ft/s [15]. The original control design in [15] involved designing a digital flight control system and synthesizing controllers for the F-14 lateral directional axis during powered approach using the structured singular-value μ framework. The controllers were designed for an angle of attack, α, of 10.5 deg and an airspeed of 137 Kn. In this work, a discrete time sliding mode tracking controller will be designed for the aforementioned described flight conditions, where it is assumed that measurements from a key sensor have been lost.

The paper has been organized as follows. The system description and the problem formulation is first given, followed by the hyperplane design and the stability analysis of the sliding dynamics. A control law with a feedback and a feedforward term will be introduced. The closed loop stability analysis and the reachability problem will then be discussed for the resulting error dynamics. Finally, the F-14 model dynamics and the fault tolerant control design for the F-14 aircraft will be described.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider the discrete, linear, time invariant state space system representation with uncertainties as given as follows:

$$x_{p_{k+1}} = A_p x_{p_k} + B_p u_{p_k} + B_{p_d} d_{p_k} \tag{1}$$

$$y_{p_k} = [(y_{p_1})_k \dots (y_{p_p})_k]^T = C_p x_{p_k}, (y_{p_i})_k = C_{p_i} x_{p_k} \tag{2}$$

where $x_{p_k} \in \mathbb{R}^n$ is the state vector, $y_{p_k} \in \mathbb{R}^p$ is the output vector, $u_{p_k} \in \mathbb{R}^m$ is the control input, and $d_{p_k} \in \mathbb{R}^q$ is the disturbance. It is assumed that the pair (A_p, B_p) is controllable and, without loss of generality, that $\text{rank}(C_p) = p$ and that column $\text{rank}(B_p) = m$. It is assumed that the matrix A is invertible. It is not assumed that the usual matching condition $rank(B_p|B_{p_d}) = rank(B_p)$

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is satisfied. Let the disturbance d_{p_k} be a bounded disturbance such that $\|d_{p_k}\| < \rho_1$. The nominal part of the plant (1)–(2) is defined as the representation when the disturbance $d_k = 0$. The aim here is to design an integral action tracking control using the extended outputs. The output vector y_{p_k} will now be partitioned as

$$y_{p_k} = \begin{bmatrix} y_{p_{m_k}} \\ y_{p_{(p-m)_k}} \end{bmatrix} \tag{3}$$

where $y_{p_{m_k}} \in \mathbb{R}^m$ are the outputs for which a tracking control scheme will required to be developed and $y_{p_{(p-m)}} \in \mathbb{R}^{(p-m)}$ represent the rest of the sensor measurements that will not be tracked but will be available for feedback. The plant output matrix in this case can then be correspondingly partitioned as follows:

$$C_p = \begin{bmatrix} C_{p_m} \\ C_{p_{(p-m)}} \end{bmatrix} \tag{4}$$

with $C_{p_m} \in \mathbb{R}^{m \times n}$ and $C_{p_{(p-m)}} \in \mathbb{R}^{(p-m) \times n}$. Now, introduce the integral action states $x_r \in \mathbb{R}^m$, which satisfy

$$x_{r_{k+1}} = x_{r_k} + (r_k - y_{p_{m_k}}) \tag{5}$$

where the signal $r(k) \in \mathbb{R}^m$ is the reference signal. The states of the system (1) are augmented with the integral action states to obtain

$$x_k = \begin{bmatrix} x_{r_k} \\ x_{p_k} \end{bmatrix} \tag{6}$$

Using (1)–(2) and (5), the augmented system is found to be

$$x_{k+1} = Ax_k + Bu_k + B_d d_k + B_r r_k \tag{7}$$

$$y_k = Cx_k \tag{8}$$

with $x_k \in \mathbb{R}^{n+m}$ and $y_k = col(x_{r_k}, y_{p_k})$. The augmented system matrices are

$$A = \begin{bmatrix} I_m & -C_{p_m} \\ 0 & A_p \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ B_p \end{bmatrix}, \; B_d = \begin{bmatrix} 0 \\ B_{p_d} \end{bmatrix}$$

$$B_r = \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} I_m & 0 \\ 0 & C_p \end{bmatrix} \tag{9}$$

The following lemma can then be stated to show the relationship between the invariant zeros of the triple (A_p, B_p, C_p) and (A, B, C).

Lemma 2.1
The invariant zeros of the triple (A_p, B_p, C_p) are the invariant zeros of the triple (A, B, C)

Proof
Consider the triple (A_p, B_p, C_p) with the assumption that $\text{rank}(B_p) = m$, $\text{rank}(C_p) = p$, and $\text{rank}(C_p B_p) = m$. Assume that the pair (A_p, B_p) is controllable and the pair (A_p, C_p) is detectable. Introduce a coordinate transformation such that the triple (A_p, B_p, C_p) has the form

$$A_p = \begin{bmatrix} A_{p_{11}} & A_{p_{12}} \\ A_{p_{21}} & A_{p_{22}} \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ B_{p_2} \end{bmatrix}, \quad C_p = \begin{bmatrix} 0 & T \end{bmatrix} \tag{10}$$

and where the matrix $A_{p_{11}} \in \mathbb{R}^{n-m}$ and has the following structure

$$A_{p_{11}} = \left[\begin{array}{cc|c} A^o_{p_{11}} & A^o_{p_{12}} & \multirow{2}{*}{$A^m_{p_{12}}$} \\ 0 & A^o_{p_{22}} & \\ \hline 0 & A^o_{p_{21}} & A^m_{p_{22}} \end{array}\right] \tag{11}$$

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where $A^o_{p_{11}} \in \mathbb{R}^{r\times r}$, $A^o_{p_{22}} \in \mathbb{R}^{(n-p-r)\times(n-p-r)}$, and $A^o_{p_{21}} \in \mathbb{R}^{(p-m)\times(n-p-r)}$ for some $r > 0$ and the pair $\left(A^o_{p_{22}}, A^o_{p_{21}}\right)$ is completely observable. In the aforementioned coordinate system, the eigenvalues of $A^o_{p_{11}}$ are the invariant zeros of the triple $(A_p,\ B_p,\ C_p)$ [11]. Now, consider the Rosenbrock matrix for the augmented triple $(A,\ B,\ C)$

$$G(z) = \begin{bmatrix} zI - A & B \\ -C & 0 \end{bmatrix} \tag{12}$$

Substituting for A_p, B_p and C_p in (12) and noticing that if $z \in \mathbb{C}$ is an invariant zero of the triple $(A,\ B,\ C)$, then the Rosenbrock system matrix $G(z)$ defined in (12) should lose normal rank at the frequency z, gives

$$\operatorname{rank}(G(z)) = \operatorname{rank}\begin{bmatrix} zI_m - I_m & C_{pm} & 0 \\ 0 & zI_n - A_p & B_{p_2} \\ I_m & 0 & 0 \\ 0 & -C_p & 0 \end{bmatrix} \tag{13}$$

$$= \operatorname{rank}\begin{bmatrix} zI_m - I_m & C_{pm} \\ 0 & [zI_{(n-m)} - A_{p_{11}} \ \ -A_{p_{12}}] \\ I_m & 0 \\ 0 & -\begin{bmatrix} C_{pm} \\ C_{p(p-m)} \end{bmatrix} \end{bmatrix} + m \tag{14}$$

$$= \operatorname{rank}\begin{bmatrix} zI_m - I_m & 0 \\ 0 & [zI_{(n-m)} - A_{p_{11}} \ \ -A_{p_{12}}] \\ I_m & 0 \\ 0 & -\begin{bmatrix} C_{pm} \\ C_{p(p-m)} \end{bmatrix} \end{bmatrix} + m \tag{15}$$

$$\Longrightarrow\ G(z) \text{ loses rank } \iff \begin{bmatrix} zI_{(n-m)} - A_{p_{11}} \ \ -A_{p_{12}} \\ -\begin{bmatrix} C_{pm} \\ C_{p(p-m)} \end{bmatrix} \end{bmatrix} \text{loses rank} \tag{16}$$

consider the rank of the matrix

$$\begin{bmatrix} zI_{(n-m)} - A_{p_{11}} \ \ -A_{p_{12}} \\ -\begin{bmatrix} C_{pm} \\ C_{p(p-m)} \end{bmatrix} \end{bmatrix} \tag{17}$$

Let the matrix T in (10) be partitioned as

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \tag{18}$$

where $T_1 \in \mathbb{R}^{(p-m)\times m}$ and $T_2 \in \mathbb{R}^{m\times m}$. Utilizing the structure of C_p from (10) with the matrix T partitioned as in (18), in (17) gives

$$\begin{bmatrix} zI_{(n-m)} - A_{p_{11}} & -A_{p_{12}} \\ \begin{bmatrix} 0 & -T_1 \end{bmatrix} & -T_2 \end{bmatrix} \tag{19}$$

Substituting for $A_{p_{11}}$ from (11) and repartitioning gives

$$\Longrightarrow \operatorname{rank}\begin{bmatrix} zI_{(n-m)} - A_{p_{11}} & -A_{p_{12}} \\ \begin{bmatrix} 0 & -T_1 \end{bmatrix} & -T_2 \end{bmatrix} = \operatorname{rank}\left[\begin{array}{cc|c} zI - A^o_{p_{11}} & -A^o_{p_{12}} & \\ 0 & zI - A^o_{p_{22}} & \star \\ 0 & -A^o_{p_{21}} & \\ \hline 0 & 0 & -T \end{array}\right] \tag{20}$$

where the subblock $\star$ represents a matrix that would not be required for further analysis. The matrix T is full rank, and $G(z)$ will lose rank if and only if

$$\begin{bmatrix} zI - A^o_{p_{11}} & -A^o_{p_{12}} \\ 0 & zI - A^o_{p_{22}} \\ 0 & -A^o_{p_{21}} \end{bmatrix} \text{loses rank} \tag{21}$$

By construction, the pair $\left(A^o_{p_{22}}, A^o_{p_{21}}\right)$ is observable, and hence, from the Popov Belevitch Hautus test, it can be checked that the rank

$$\begin{bmatrix} zI - A^o_{p_{22}} \\ -A^o_{p_{21}} \end{bmatrix} = n - p - r \quad \forall\, z \in \mathbb{C} \tag{22}$$

Hence, $G(z)$ loses rank implies $\det\left(zI - A^o_{p_{11}}\right) = 0$. This implies that the eigenvalues of $A^o_{p_{11}}$ are the invariant zeros of the triple $(A,\ B,\ C)$. As stated before, the eigenvalues of $A^o_{p_{11}}$ are also the invariant zeros of $(A_p,\ B_p,\ C_p)$. Hence, any invariant zero of the triple $(A_p,\ B_p,\ C_p)$ will also be an invariant zero of the triple $(A,\ B,\ C)$. □

Remark 1
The algorithm proposed in this paper is applicable for underactuated systems and for square systems where the condition $\text{rank}(C_p B_p) = m$ holds. Thus, the condition $\text{rank}(C_p B_p) = m$ assumed in Lemma 2.1 is a necessary condition. The proposed tracking algorithm in this paper is thus not intended for use with overactuated systems.

An extended output matrix $\tilde{C}$ is then constructed using only the plant outputs, without any a priori assumptions on the system (7)–(8) relating to the stability of the invariant zeros:

$$\tilde{C} = \begin{bmatrix} C_1 \\ \vdots \\ C_m \\ C_{m+1} \\ \vdots \\ C_{m+p} \\ \vdots \\ \left(C_{m+1} A^{-\mu_{m+1}+1}\right) \\ \vdots \\ \left(C_{m+p} A^{-\mu_{p+m}+1}\right) \end{bmatrix} \tag{23}$$

such that $\tilde{C}$ is full rank, $\text{rank}(\tilde{C}B) = \text{rank}(B)$, and any invariant zeros of the triple $(A, B, \tilde{C})$ lie inside the unit disk. Also, the μ_i are chosen such that $\tilde{p} = m + \sum_{i=m+1}^{m+p} \mu_i$ is minimal.

Remark 2
Note that the work proposed here parallels in some respect the work in [17] where fast output sampling is used for designing discrete time sliding mode controllers. However, in [17], the output is extended to the dimension of the state, and the invertibility property of the sampled system is not employed. In this work, the output is extended so that the core triple used for sliding surface design is output feedback stabilizable. This is also achieved without taking additional samples of the measured output signals; instead, both current and past output measurements are used by the proposed discrete time sliding mode controller.

It is now important to show the relation between the invariant zeros of the original system triple (A, B, C) and the invariant zeros of the augmented system triple $(A, B, \tilde{C})$. It has been shown that any invariant zeros of the triple $(A, B, \tilde{C})$ are amongst the invariant zeros of the triple

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(A, B, C) [13]. It has also been shown that if an appropriate choice of extended outputs is available, then the invariant zeros from the original triple (A, B, C) may disappear from the augmented system $(A, B, \tilde{C})$ [13]. This is particularly useful if any of the invariant zeros of (A, B, C) are unstable.

3. HYPERPLANE DESIGN AND STABILITY ANALYSIS

Now, assume that it is possible to choose $\tilde{C}$ as stated in (23). The problem now becomes one of finding a suitable sliding variable s_k that is a function of the augmented outputs only. Choose a set of past plant outputs along with the present plant outputs and form a new extended output vector $\tilde{y}_k$ as shown as follows:

$$\tilde{y}_k = \begin{bmatrix} (y_1)_k \\ \vdots \\ (y_m)_k \\ (y_{m+1})_k \\ \vdots \\ (y_{m+p})_k \\ \vdots \\ (y_{m+1})_{k-\mu_{m+1}+1} \\ \vdots \\ (y_{m+p})_{k-\mu_{m+p}+1} \end{bmatrix} \tag{24}$$

From the system (1)–(2), the aforementioned vector $\tilde{y}_k$, can be computed as follows:

$$\tilde{y}_k = \tilde{C}x_k - \begin{bmatrix} 0_{(p+m)\times l_1} \\ \mathbf{M}_{(\tilde{p}-(p+m))\times l_1} \end{bmatrix} U_k - \begin{bmatrix} 0_{(p+m)\times q_1} \\ \mathbf{M}_{\mathbf{d}(\tilde{p}-(p+m))\times l_1} \end{bmatrix} D_k - \begin{bmatrix} 0_{(p+m)\times l_1} \\ \mathbf{M}_{\mathbf{r}(\tilde{p}-(p+m))\times l_1} \end{bmatrix} R_k$$

with $l_1 = m(\tilde{p} - (p + m))$ and $q_1 = q(\tilde{p} - (p + m))$. The vectors $U_k \in \mathbb{R}^{l_1}$, $R_k \in \mathbb{R}^{l_1}$, and $D_k \in \mathbb{R}^{q_1}$ being

$$U_k = \begin{bmatrix} \bar{u}_1^T & \bar{u}_2^T & \cdots & \bar{u}_p^T \end{bmatrix}^T, D_k = \begin{bmatrix} \bar{d}_1^T & \bar{d}_2^T & \cdots & \bar{d}_p^T \end{bmatrix}^T,$$
$$R_k = \begin{bmatrix} \bar{r}_1^T & \bar{r}_2^T & \cdots & \bar{r}_p^T \end{bmatrix}^T \tag{25}$$

and for $i = 1, \ldots, p$,

$$\bar{u}_i = \begin{bmatrix} u_{k-1}^T & u_{k-2}^T & \cdots & u_{k-\mu_i+1}^T \end{bmatrix}^T \tag{26}$$

$$\bar{r}_i = \begin{bmatrix} r_{k-1}^T & r_{k-2}^T & \cdots & r_{k-\mu_i+1}^T \end{bmatrix}^T \tag{27}$$

and

$$\bar{d}_i = \begin{bmatrix} d_{k-1}^T & d_{k-2}^T & \cdots & d_{k-\mu_i+1}^T \end{bmatrix}^T \tag{28}$$

The matrix $\mathbf{M}$ is block diagonal and has the form

$$\mathbf{M} = \mathrm{diag}\{\mathbf{M}_1, \ldots, \mathbf{M}_p\},$$

where

$$\mathbf{M}_i = \begin{pmatrix} C_i A^{-1} B & 0 & \ldots & \ldots & 0 \\ C_i A^{-2} B & C_i A^{-1} B & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & C_i A^{-1} B & 0 \\ C_i A^{-\mu_i+1} B & \ldots & \ldots & C_i A^{-2} B & C_i A^{-1} B \end{pmatrix}$$

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with $\mathbf{M}_i \in \mathbb{R}^{(\mu_i-1)\times(m(\mu_i-1))}$. Similarly, the block diagonal matrix $\mathbf{M_r}$ is

$$\mathbf{M_r} = \text{diag}\{\mathbf{M_{r}}_1, \ldots, \mathbf{M_{r}}_p\}$$

$$\mathbf{M_r}_i = \begin{pmatrix} C_i A^{-1} B_r & 0 & \ldots & \ldots & 0 \\ C_i A^{-2} B_r & C_i A^{-1} B_r & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & C_i A^{-1} B_r & 0 \\ C_i A^{-\mu_i+1} B_r & \ldots & \ldots & C_i A^{-2} B_r & C_i A^{-1} B_r \end{pmatrix}$$

with $\mathbf{M_r}_i \in \mathbb{R}^{(\mu_i-1)\times(m(\mu_i-1))}$ and the block diagonal matrix $\mathbf{M_d}$ having the form

$$\mathbf{M_d} = \text{diag}\left\{\mathbf{M_d}_1, \ldots, \mathbf{M_d}_p\right\}$$

$$\mathbf{M_d}_i = \begin{pmatrix} C_i A^{-1} B_d & 0 & \ldots & \ldots & 0 \\ C_i A^{-2} B_d & C_i A^{-1} B_d & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & C_i A^{-1} B_d & 0 \\ C_i A^{-\mu_i+1} B_d & \ldots & \ldots & C_i A^{-2} B_d & C_i A^{-1} B_d \end{pmatrix}$$

and $\mathbf{M_d}_i \in \mathbb{R}^{(\mu_i-1)\times(q(\mu_i-1))}$.

Defining the sliding manifold in terms of known variables gives the following:

$$\begin{aligned} s_k &= F \tilde{y}_k + \Gamma_u U_k + \Gamma_r R_k & (29) \\ &= F \tilde{C} x_k - \Gamma_d D_k, & (30) \end{aligned}$$

where the design matrix $F \in \mathbb{R}^{m\times\tilde{p}}$ and with

$$\Gamma_u = F \begin{bmatrix} 0_{(p+m)\times l_1} \\ \mathbf{M}_{(\tilde{p}-(p+m))\times l_1} \end{bmatrix}, \; \Gamma_r = F \begin{bmatrix} 0_{(p+m)\times l_1} \\ \mathbf{M}_{\mathbf{r}(\tilde{p}-(p+m))\times l_1} \end{bmatrix}$$

and

$$\Gamma_d = F \begin{bmatrix} 0_{(p+m)\times q_1} \\ \mathbf{M}_{\mathbf{d}(\tilde{p}-(p+m))\times q_1} \end{bmatrix} \quad (31)$$

The stability of the resulting sliding motion can be analyzed by introducing a coordinate transformation $x \rightarrow \hat{T} x = \hat{x}$ with the following structure:

$$\begin{aligned} &\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \; \hat{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \; \hat{B}_d = \begin{bmatrix} \hat{B}_{d_1} \\ \hat{B}_{d_2} \end{bmatrix}, \hat{B}_r = \begin{bmatrix} \hat{B}_{r_1} \\ \hat{B}_{r_2} \end{bmatrix} \\ &\hat{C} = \begin{bmatrix} 0 & T \end{bmatrix}, \quad (32) \end{aligned}$$

where $T \in \mathbb{R}^{\tilde{p}\times\tilde{p}}$ is an orthogonal matrix. The matrix $\hat{A}_{11} \in \mathbb{R}^{n\times n}$ and has the following structure

$$\hat{A}_{11} = \left[\begin{array}{cc|c} \hat{A}_{11}^o & \hat{A}_{12}^o & \multirow{2}{*}{$\hat{A}_{12}^m$} \\ 0 & \hat{A}_{22}^o & \\ \hline 0 & \hat{A}_{21}^o & \hat{A}_{22}^m \end{array}\right]$$

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with $\hat{A}^o_{11} \in \mathbb{R}^{r_1 \times r_1}$, $\hat{A}^o_{22} \in \mathbb{R}^{(n+m-\tilde{p}-r_1)\times(n+m-\tilde{p}-r_1)}$, and $\hat{A}^o_{21} \in \mathbb{R}^{(\tilde{p}-m)\times(n+m-\tilde{p}-r_1)}$ for some $r_1 \geqslant 0$, and the pair $\left(\hat{A}^o_{22}, \hat{A}^o_{21}\right)$ is completely observable. The remaining subblocks in the system matrix are partitioned accordingly. The corresponding switching surface matrix is given by

$$\left[\begin{array}{cc} \overset{\tilde{p}-m}{\overset{\leftrightarrow}{F_1}} & \overset{m}{\overset{\leftrightarrow}{F_2}} \end{array}\right] = FT, \tag{33}$$

where T is the matrix from (32). As a result,

$$F\hat{C} = \left[\begin{array}{cc} F_1 C_f & F_2 \end{array}\right] \tag{34}$$

where

$$C_f = \left[\begin{array}{cc} 0_{(\tilde{p}-m)\times(n+m-\tilde{p})} & I_{(\tilde{p}-m)} \end{array}\right] \tag{35}$$

Therefore, $F\hat{C}\hat{B} = F_2 B_2$ and the square matrix F_2 is nonsingular. The disturbance distribution matrix Γ_d in the new coordinates $\hat{\Gamma}_d$ will be of the form

$$\left[\begin{array}{cc} \overset{q_1-m}{\overset{\leftrightarrow}{\hat{\Gamma}_{d_1}}} & \overset{m}{\overset{\leftrightarrow}{\hat{\Gamma}_{d_2}}} \end{array}\right] = \hat{\Gamma}_d \tag{36}$$

Consider the nominal plant for the system (1)–(2) when the reference signal $r_k \neq 0$ and when $d_k = 0$ in the new coordinate system. Hence, on the sliding surface when $s_k = 0$, it can be seen that

$$(\hat{x}_2)_k = F_2^{-1}(-F_1 C_f (\hat{x}_1)_k) \tag{37}$$

Using the result from (37) in the first equation of the canonical form in (32), the following can now be computed:

$$(\hat{x}_1)_{k+1} = \left(\hat{A}_{11} - \hat{A}_{12} F_2^{-1} F_1 C_f\right)(\hat{x}_1)_k + \hat{B}_{r_1}\hat{r}_k. \tag{38}$$

The solution for the aforementioned equation is as follows:

$$(\hat{x}_1)_{k+1} = \left(\hat{A}^s_{11}\right)^k (\hat{x}_1)_o + \sum_{j=0}^{k-1}\left(\hat{A}^s_{11}\right)^{k-j-1}\hat{B}_{r_1}\hat{r}_j. \tag{39}$$

If $\hat{A}^s_{11}$ is designed by the choice of F_1 such that it has eigenvalues within the unit circle, then it can be shown that

$$\|\hat{A}^s_{11}\| \leqslant \gamma\lambda^k, \tag{40}$$

where $\gamma > 0$ and $0 < \lambda < 1$. Also, it can further be shown that:

$$\sum_{j=0}^{k-1}\|\left(\hat{A}^s_{11}\right)^j\| \leqslant \gamma\sum_{j=0}^{k-1}\lambda^j \leqslant \gamma\sum_{j=0}^{\infty}\lambda^j \leqslant \frac{\gamma}{1-\lambda}. \tag{41}$$

If the reference $r_k = r_s$ is a constant, then it can be shown that the sliding dynamics in the presence of a reference signal will be uniformly ultimately bonded by $\frac{\gamma}{1-\lambda}\|\hat{B}_{r_1}\|\|\hat{r}_s\|$.

For the system (1)–(2) with uncertainty, the sliding surface (30) has a disturbance term acting on it explicitly. It will now be shown that the reduced order motion in the presence of bounded disturbance will be confined within a region around the sliding surface $\hat{s}$. To show this, let:

$$\hat{s}_k = 0 \tag{42}$$

so that

$$(\hat{x}_2)_k = F_2^{-1}(-F_1 C_f (\hat{x}_1)_k + \hat{\Gamma}_d \hat{D}_k). \tag{43}$$

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Substituting for $(\hat{x}_2)_k$ in $(\hat{x}_1)_{k+1}$, one can obtain

$$(\hat{x}_1)_{k+1} = \hat{A}_{11}^s (\hat{x}_1)_k + \hat{A}_{12} F_2^{-1} \hat{\Gamma}_d \hat{D}_k + \hat{B}_{d_1} \hat{d}_k + \hat{B}_{r_1} \hat{r}_k.$$

The solution for $(\hat{x}_1)_{k+1}$ is the following:

$$(\hat{x}_1)_k = \left(\hat{A}_{11}^s\right)^k (\hat{x}_1)_o + \sum_{j=0}^{k-1} \left(\hat{A}_{11}^s\right)^{k-j-1} \left(\hat{A}_{12} F_2^{-1} \hat{\Gamma}_d \hat{D}_j + \hat{B}_{d_1} \hat{d}_j + \hat{B}_{r_1} \hat{r}_j\right). \tag{44}$$

Taking norms and using the bounds from (41) in (44) gives

$$\|(\hat{x}_1)_k\| \leqslant \| \left(\hat{A}_{11}^s\right)^k \| \|(\hat{x}_1)_o\| + \| \sum_{j=0}^{k-1} \left(\hat{A}_{11}^s\right)^{k-j-1} \| \left(\|\hat{A}_{12}\| \|F_2^{-1}\| \|\hat{\Gamma}_d\| \|\hat{D}_j\| + \|\hat{B}_{d_1}\| \|\hat{d}_j\| + \|\hat{B}_{r_1}\| \|\hat{r}_j\|\right). \tag{45}$$

Let $\rho_2 = \sup_{k>0} \|\hat{D}_k\|$, for some scalar $\rho_2 > 0$. If the reference $\hat{r_j} = \hat{r}_s$ is constant and noting that $\|\hat{d}_j\| < \rho_1$, then the bounds on $\|(\hat{x}_1)_k\|$ can be written as follows:

$$\|(\hat{x}_1)_k\| \leqslant \gamma \lambda^k \|(\hat{x}_1)_o\| + \frac{\gamma}{1-\lambda} \left(\|\hat{A}_{12}\| \|F_2^{-1}\| \|\hat{\Gamma}_d\| \rho_2 + \|\hat{B}_{d_1}\| \rho_1 + \|\hat{B}_{r_1}\| \|\hat{r}_s\|\right), \tag{46}$$

and the reduced order sliding motion will be uniformly ultimately bounded by $\frac{\gamma}{1-\lambda} \left(\|\hat{A}_{12}\| \|F_2^{-1}\| \|\hat{\Gamma}_d\| \rho_2 + \|\hat{B}_{d_1}\| \rho_1 + \|\hat{B}_{r_1}\| \|\hat{r}_s\|\right)$.

From the aforementioned discussion, the following can be inferred with regard to the regulation problem, when $r_k = 0$:

- For the nominal plant (1)–(2), the reduced order sliding motion in the coordinate system (32) is governed by a free motion with system matrix

$$\hat{A}_{11}^s = \hat{A}_{11} - \hat{A}_{12} K C_f \tag{47}$$

- In the presence of the disturbance, the reduced order sliding motion will be uniformly ultimately bounded by $\frac{\gamma}{1-\lambda} \left(\|\hat{A}_{12}\| \|F_2^{-1}\| \|\hat{\Gamma}_d\| \rho_2 + \|\hat{B}_{d_1}\| \rho_1\right)$.

Note that, for the existence of a stable bounded sliding motion, $\hat{A}_{11}^s = \hat{A}_{11} - \hat{A}_{12} K C_f$ must be stable. It is well known that if the triple $\left(\hat{A}, \hat{B}, \hat{C}\right)$ has any invariant zeros, then for the output feedback case, the reduced order dynamics during sliding will have these invariant zeros appear among the poles of the closed loop dynamics; in which case, it will not be possible to arbitrarily assign the poles to obtain a K that will stabilize $\hat{A}_{11}^s = \hat{A}_{11} - \hat{A}_{12} K C_f$ [11]. In this case, a new subsystem can be constructed such that $\lambda\left(A_{11}^s\right) = \lambda\left(\hat{A}_{11}^o\right) \cup \lambda\left(\tilde{A}_{11} - \hat{A}_{122} K \tilde{C}_f\right)$. The matrix $\hat{A}_{122}$ here is defined as the bottom $(n - r_1)$ rows of the matrix $\hat{A}_{12}$, where r_1 is the number of invariant zeros in the system and the matrix $\tilde{A}_{11}$ is defined as the corresponding $(n - r_1)$ rows, and the last $(n - r_1)$ columns of the matrix $\hat{A}_{11}$. If $\tilde{C}_f$ is the last $(n - r_1)$ columns of C_f, then the triple $\left(\tilde{A}_{11}, \hat{A}_{122}, \tilde{C}_f\right)$ is both observable and controllable, and the spectrum of $\hat{A}_{11}^o$ represents the invariant zeros of the triple $\left(\hat{A}, \hat{B}, \hat{C}\right)$ [12]. Thus, for designing a sliding surface that will provide a stable sliding motion, the problem now becomes the design of a static output feedback gain K that will stabilize the triple $\left(\tilde{A}_{11}, \hat{A}_{122}, \tilde{C}_f\right)$.

To design the sliding surface, consider the method for the design of a static output feedback gain as given in [18, 19]. The technique in [18, 19] introduces slack variables to decouple the Lyapunov matrix and the static output feedback gain. With the additional slack variables and a chosen state

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space variable, an LMI problem (see for example [20]) is solved to obtain the static output feedback controller. The following theorem is required to formulate the design of the sliding surface using this method.

Theorem 3.1
Let the matrix A_o be defined as $A_o = \tilde{A}_{11} + \hat{A}_{122}K_o$. A static output feedback gain K is stabilizing for the triple $\left(\tilde{A}_{11}, \hat{A}_{122}, \tilde{C}_f\right)$ if and only if there exist a positive definite matrix $\Xi = \Xi^T > 0$ in $\mathbb{R}^{n\times n}$, nonsingular matrices $G_1 \in \mathbb{R}^{m\times m}$ and $E_4 \in \mathbb{R}^{n\times n}$, nonnull matrices $E_1 \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{m\times p}$, and arbitrary matrices $E_2 \in \mathbb{R}^{n\times n}$, $E_3 \in \mathbb{R}^{m\times n}$ such that the following LMI:

$$\begin{bmatrix} E_1A_o + A_o^TE_1^T - \Xi & * & * & * \\ E_2A_o & -\Xi & * & * \\ E_3A_o + A_{122}^TE_1^T + (L\tilde{C}_f - G_1K_o) & A_{122}^TE_2^T & E_3A_{122} + A_{122}^TE_3^T - (G_1 + G_1^T) & * \\ E_4A_o - E_1^T & \Xi - E_2^T & E_4A_{122} - E_3^T & -E_4 - E_4^T \end{bmatrix} < 0 \tag{48}$$

is feasible for a given state feedback gain K_o that stabilizes the pair $\left(\tilde{A}_{11}, \acute{A}_{122}\right)$ with the static output feedback gain given by

$$K = G_1^{-1}L. \tag{49}$$

For a conclusive proof of the aforementioned theorem, refer to [19].

In the following section, a control action with both a feedback and a feedforward terms will be developed to ensure tracking, and the resulting error dynamics will be analyzed.

4. THE FEEDFORWARD GAIN AND THE ERROR DYNAMICS

Introduce a nonsingular state transformation of the form $\hat{x} \to \bar{T}\hat{x}$ where $\bar{T}$ is given as follows:

$$\bar{T} = \begin{bmatrix} I_n & 0 \\ -KC_f & I_m \end{bmatrix} \tag{50}$$

where the matrices I_n and I_m are the identity matrices of dimension n and m, respectively, and the matrix $KC_f \in \mathbb{R}^{m\times n}$. With the aforementioned transformation, the system (1)–(2) is of the following form:

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \ \bar{B}_d = \begin{bmatrix} \bar{B}_{d_1} \\ \bar{B}_{d_2} \end{bmatrix}, \ \bar{B}_r = \begin{bmatrix} \bar{B}_{r_1} \\ \bar{B}_{r_2} \end{bmatrix}$$
$$F\bar{C} = \begin{bmatrix} 0 & F_2 \end{bmatrix}. \tag{51}$$

Here, $\bar{A}_{11} = \hat{A}_{11}^s$ with $\bar{A}_{11} \in \mathbb{R}^{n\times n}$ and the rest of the submatrices in the matrix $\bar{A}$ are conformably partitioned. Let the reference signal $r_k = r_s = const, \ \forall\ k > k_s$. The control law u_k is chosen such that it has a feed forward term $F_r \in \mathbb{R}^{m\times m}$, and a feedback term is given as follows:

$$u_k = -(F\tilde{C}A^{-1}B)s_k + F_r r_k. \tag{52}$$

In the new coordinates, the control law is the following:

$$\bar{u}_k = -(F\bar{C}\bar{A}^{-1}\bar{B})\bar{s}_k + F_r\bar{r}_k. \tag{53}$$

The feedforward gain matrix F_r is unknown and has to be determined. Let the steady state value of the states $\bar{x}_s$ be

$$\bar{x}_s = (I - \bar{A}_{cl})^{-1}(\bar{B}_r + \bar{B}F_r)\bar{r}_s, \tag{54}$$

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where the matrix $\bar{A}_{cl}$ is the closed loop state matrix and is stable by design. The matrix $(I - \bar{A}_{cl})$ is invertible and well defined because by design, the eigenvalues of $\bar{A}_{cl}$ are inside the unit disk. Define the error of the states and the steady state values of the states as $\bar{e}_k = \bar{x}_k - \bar{x}_s$. Then, $\bar{e}_{k+1}$ can be computed as follows:

$$\bar{e}_{k+1} = \bar{A}_{cl}\bar{e}_k + \bar{B}_D\bar{w}_k, \tag{55}$$

where the disturbance input $\bar{w}_k \in \mathbb{R}^{(q+q_1)}$ and is such that $\bar{w}_k = \begin{bmatrix} \bar{d}_k^T & \bar{D}_k^T \end{bmatrix}^T$ and the matrix

$$\bar{B}_D = \bar{B}(F\bar{C}\bar{A}^{-1}\bar{B})^{-1}[0 \;\; \bar{\Gamma}_d] + \bar{B}_w.$$

The matrix $\bar{B}_w \in \mathbb{R}^{(n+m)\times(q+q_1)}$ with $\bar{B}_w = [\bar{B}_d \;\; 0]$. As can be seen because the matrix $\bar{A}_{cl}$ is stable, in the absence of uncertainty $\bar{e}_k \to 0$ as $k \to \infty$. This implies that because steady state is achieved, it follows that from the first m equations in (7) that $\bar{y}_{p_k} = \bar{r}_s$, and hence, tracking is achieved. Now, consider

$$F\bar{C}\bar{x}_s = F\bar{C}(I - \bar{A}_{cl})^{-1}(\bar{B}_r + \bar{B}F_r)\bar{r}_s.$$

If $F\bar{C}\bar{x}_s = 0$, then it can be computed that

$$F_r = -(F\bar{C}(I - \bar{A}_{cl})^{-1}B)^{-1}F\bar{C}(I - \bar{A}_{cl})^{-1}\bar{B}_r.$$

Hence, the control law $\bar{u}_k$ can be written as follows:

$$\bar{u}_k = -(F\bar{C}\bar{A}^{-1}\bar{B})\bar{s}_k - (F\bar{C}(I - \bar{A}_{cl})^{-1}\bar{B})^{-1}F\bar{C}(I - \bar{A}_{cl})^{-1}\bar{B}_r\bar{r}_k. \tag{56}$$

Now, define a function $\bar{s}_{e_k} \in \mathbb{R}^m$ such that

$$\bar{s}_{e_k} = \bar{s}_k - \bar{s}_{x_s}, \tag{57}$$

where $\bar{s}_{x_s} \in \mathbb{R}^m$ is defined as follows:

$$\bar{s}_{x_s} = F\bar{C}\bar{x}_s \tag{58}$$

and with $\bar{s}_k$ as defined in (30). A sliding surface that is a function of the error dynamics $\bar{e}_k$ and the disturbance, can then be defined as

$$\bar{s}_{e_k} = F\bar{C}\bar{e}_k - \bar{\Gamma}_d\bar{D}_k. \tag{59}$$

Define a new output variable

$$\begin{aligned} \bar{Y}_k &= \bar{\bar{y}}_k - \begin{bmatrix} 0_{(m+p)\times l_1} \\ \bar{\mathbf{M}}_{(\tilde{p}-(p+m))\times l_1} \end{bmatrix} \bar{U}_k \\ &= \bar{C}\bar{x}_k - \bar{\Gamma}_1\bar{w}_k \end{aligned} \tag{60}$$

with $\bar{Y}_k \in \mathbb{R}^{\tilde{p}}$ and where the matrix $\bar{\Gamma}_1$ is such that

$$\bar{\Gamma}_1 = \begin{bmatrix} 0_{(m+p)\times q} & 0_{(m+p)\times q_1} \\ 0_{(\tilde{p}-(m+p))\times q} & \bar{\mathbf{M}}_{\mathbf{d}(\tilde{p}-(m+p))\times q_1} \end{bmatrix}$$

Again, define $\bar{Y}_s = \bar{C}x_s$ and define the error

$$e_{\bar{Y}_k} = \bar{Y}_k - \bar{Y}_s. \tag{61}$$

Then, the following error system can be formed

$$\bar{e}_{k+1} = \bar{A}_{cl}\bar{e}_k + \bar{B}_D\bar{w}_k \tag{62}$$

$$e_{\bar{Y}_k} = \bar{C}\bar{e}_k - \bar{\Gamma}_1\bar{w}_k. \tag{63}$$

The system (62)–(63) will now be considered for the stability analysis.

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4.1. *Closed loop analysis and reachability problem*

Consider the following system formed from the error dynamics (62)–(63) along with $\bar{z}_{e_k}$, a performance output:

$$\bar{e}_{k+1} = \bar{A}_{cl}\bar{e}_k + \bar{B}_D\bar{w}_k \quad (64)$$

$$e_{\bar{Y}_k} = \bar{C}\bar{e}_k - \bar{\Gamma}_1\bar{w}_k \quad (65)$$

$$\bar{z}_{e_k} = \begin{bmatrix} \bar{e}_{\bar{Y}_k} \\ \bar{s}_{e_k} \end{bmatrix}. \quad (66)$$

The matrix $\bar{A}_{cl}$ is stable by design. The transfer function of the closed loop system with the disturbance input $\bar{w}_k$ and the performance output $\bar{z}_k$ is given as

$$T_{\bar{z}\bar{w}} = \begin{bmatrix} I_{\bar{p}} \\ F \end{bmatrix} \left(\bar{C}(zI - \bar{A}_{cl})^{-1}\bar{B}_D - \bar{\Gamma}_1\right).$$

The objective here is to minimize the H_2 norm $\|T_{\bar{z}\bar{w}}\|_2$ of the closed loop system with the disturbance input $\bar{w}_k$ and performance output $\bar{z}_k$ such that the $\|T_{\bar{z}\bar{w}}\|_2$ is less than some prescribed positive value. Let $\bar{P} \in \mathbb{R}^{(n+m)\times(n+m)}$ be a positive definite matrix that satisfies

$$0 = \bar{P} - \bar{A}_{cl}^T\bar{P}\bar{A}_{cl} - \bar{C}^T\bar{C} - \bar{C}^T F^T F\bar{C}. \quad (67)$$

Then, the norm $\|T_{\bar{z}\bar{w}}\|_2$ can be computed as

$$\|T_{\bar{z}\bar{w}}\|_2^2 = \text{trace}\left(\bar{B}_D^T\bar{P}\bar{B}_D + \bar{\Gamma}_1^T\bar{\Gamma}_1 + \bar{\Gamma}_1^T F^T F\bar{\Gamma}_1\right). \quad (68)$$

The following theorem can now be stated to compute the H_2 norm for $T_{\bar{z}\bar{w}}$.

Theorem 4.1
The system (64)–(66) is stable with $\|T_{\bar{z}\bar{w}}\|_2 < \delta$, with $\delta > 0$, if and only if there exists positive definite matrices $\bar{P} \in \mathbb{R}^{(n+m)\times(n+m)}$, $\bar{W} \in \mathbb{R}^{(q+q_1)\times(q+q_1)}$, and a matrix $F_2 \in \mathbb{R}^{m\times m}$ such that the following set of LMI are satisfied

$$\text{trace}(\bar{W}) < \delta, \begin{bmatrix} \bar{P} & \bar{P}\bar{B}_D & 0 \\ \bar{B}_D^T\bar{P} & \bar{W} - \bar{\Gamma}_1^T\bar{\Gamma}_1 & F^T\bar{\Gamma}_1^T \\ 0 & \bar{\Gamma}_1 F & I \end{bmatrix} > 0 \quad (69)$$

$$\begin{bmatrix} \bar{P} & \bar{P}\bar{A}_{cl} & 0 \\ \bar{A}_{cl}^T\bar{P} & \bar{P} - \bar{C}_1^T\bar{C}_1 & F^T\bar{C}^T \\ 0 & \bar{C}F & I \end{bmatrix} > 0. \quad (70)$$

The first inequality can be obtained by letting (68) less than some positive definite matrix $\bar{W}$ and then applying the Schur complement. The second inequality can be obtained by replacing the equality sign with an inequality in (67) and then applying the Schur complement. Note that the matrix F_2 here is unknown. To compute the matrix F_2, solve the set of LMIs (69)–(70) with $\bar{P}$, $\bar{W}$, and F_2 as the decision variables. The matrix $\bar{P}$ obtained by solving for the LMIs (69)–(70) will be used to compute an invariant set for the trajectories of the error for the closed loop system (64)–(65). The following lemma will now be stated without proof.

Theorem 4.2
Let $\rho > 0$ and let the set $\xi(\bar{P}, \rho)$ be defined as

$$\xi(\bar{P}, \rho) = \left\{\bar{e}_k \in \mathbb{R}^{n+m} : \bar{e}_k^T\bar{P}\bar{e}_k < \rho\right\}.$$

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Then, the error dynamics of the system (64)–(65) are globally uniformly ultimately bounded by $\xi(\bar{P}, \rho)$ with

$$\rho = \epsilon_1 + \left(\|\bar{P}(I_n - (1+\alpha)\bar{P}^{-1}\bar{A}_{cl}^T \bar{P} \bar{A}_{cl})\|\right)^{-1} \left(\frac{1+\alpha}{\alpha}\bar{\lambda}\left(\bar{B}_D^T \bar{P} \bar{B}_D\right)\right) \|\bar{P}\| \rho_3^2 \tag{71}$$

for some scalar $\alpha > 0$, $\rho_3 > 0$ and for some scalar $\epsilon_1 > 0$, with ϵ_1 arbitrarily small.

Proof
Construct the Lyapunov function

$$\bar{V}_k = \bar{e}_k^T \bar{P} \bar{e}_k. \tag{72}$$

The forward difference for the aforementioned Lyapunov function is the following:

$$\Delta \bar{V}_k = \bar{V}_{k+1} - \bar{V}_k \tag{73}$$

$$\Delta \bar{V}_k = \left(\bar{A}_{cl}\bar{e}_k + \bar{B}_D \bar{w}_k\right)^T \bar{P}\left(\bar{A}_{cl}\bar{e}_k + \bar{B}_D \bar{w}_k\right) - \bar{e}_k^T \bar{P} \bar{e}_k. \tag{74}$$

Using the inequality $(a+b)^T(a+b) \leqslant (1+\alpha)a^T a + \left(1 + \frac{1}{\alpha}\right) b^T b$ for some scalar $\alpha > 0$, in (74) gives

$$\leqslant (1+\alpha)\bar{e}_k^T \bar{A}_{cl}^T \bar{P} \bar{A}_{cl}\bar{e}_k + \left(1 + \frac{1}{\alpha}\right)\bar{w}_k^T \bar{B}_D^T \bar{P} \bar{B}_D \bar{w}_k - \bar{e}_k^T \bar{P} \bar{e}_k. \tag{75}$$

Now, using the standard definition of a supremum, define ρ_3 as $\rho_3 = \underset{k>0}{sup}\, \|\bar{w}_k\|$ and let $\bar{\lambda}\left(\bar{B}_D^T \bar{P} \bar{B}_D\right)$ be the maximum eigenvalue of $\bar{B}_D^T \bar{P} \bar{B}_D$. Consider the condition when $\Delta V_k < 0$,

$$(1+\alpha)\bar{e}_k^T \bar{A}_{cl}^T \bar{P} \bar{A}_{cl}\bar{e}_k + \left(\frac{1+\alpha}{\alpha}\right)\bar{\lambda}\left(\bar{B}_D^T \bar{P} \bar{B}_D\right)\rho_3^2 - \bar{e}_k^T \bar{P} \bar{e}_k < 0 \tag{76}$$

$$(1+\alpha)\bar{e}_k^T \bar{A}_{cl}^T \bar{P} \bar{A}_{cl}\bar{e}_k + \left(\frac{1+\alpha}{\alpha}\right)\bar{\lambda}\left(\bar{B}_D^T \bar{P} \bar{B}_D\right)\rho_3^2 < \bar{e}_k^T \bar{P} \bar{e}_k \tag{77}$$

$$\Rightarrow \left(\frac{1+\alpha}{\alpha}\bar{\lambda}\left(\bar{B}_D^T \bar{P} \bar{B}_D\right)\right)\rho_3^2 < \bar{e}_k^T\left(\bar{P} - (1+\alpha)\bar{A}_{cl}^T \bar{P} \bar{A}_{cl}\right)\bar{e}_k \tag{78}$$

$$\left(\frac{1+\alpha}{\alpha}\bar{\lambda}\left(\bar{B}_D^T \bar{P} \bar{B}_D\right)\right)\rho_3^2 < \bar{e}_k^T \bar{P}\left(I_n - (1+\alpha)\bar{P}^{-1}\bar{A}_{cl}^T \bar{P} \bar{A}_{cl}\right)\bar{e}_k \tag{79}$$

$$\left(\frac{1+\alpha}{\alpha}\bar{\lambda}\left(\bar{B}_D^T \bar{P} \bar{B}_D\right)\right)\rho_3^2 < \|\bar{P}\left(I_n - (1+\alpha)\bar{P}^{-1}\bar{A}_{cl}^T \bar{P} \bar{A}_{cl}\right)\|\|\bar{e}_k\|^2 \tag{80}$$

$$\Rightarrow \|\bar{e}_k\|^2 \geqslant \|\bar{P}\left(I_n - (1+\alpha)\bar{P}^{-1}\bar{A}_{cl}^T \bar{P} \bar{A}_{cl}\right)\|^{-1}\left(\frac{1+\alpha}{\alpha}\bar{\lambda}\left(\bar{B}_D^T \bar{P} \bar{B}_D\right)\right)\rho_3^2. \tag{81}$$

Hence, depending on the initial condition $\bar{e}_0$, the error dynamics will ultimately enter an ellipsoid and thereafter will remain within the ellipsoid $\xi(P, \rho)$ in a finite time k with

$$\rho = \epsilon_1 + \left(\|\bar{P}\left(I_n - (1+\alpha)\bar{P}^{-1}\bar{A}_{cl}^T \bar{P} \bar{A}_{cl}\right)\|\right)^{-1}\left(\frac{1+\alpha}{\alpha}\bar{\lambda}\left(\bar{B}_D^T \bar{P} \bar{B}_D\right)\right)\|\bar{P}\|\rho_3^2,$$

where $\epsilon_1 > 0$ and with ϵ_1 arbitrarily small. Hence, the claim. □

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Remark 3
The matrix F_2 does not affect the control $\bar{u}_k$.

Remark 4
The closed loop state space matrix for the nominal system (51) can be computed as follows:

$$\bar{A}_{cl} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12} \end{bmatrix}. \tag{82}$$

Remark 5
The closed loop dynamics for the nominal plant (64)–(65) when $\bar{w}_k = 0$ is a reduced order dynamics. Introduce a new change of coordinates of the form $\bar{e}_k \rightarrow T_c\bar{e}_k = e_k^c$, where T_c has the form

$$T_c = \begin{bmatrix} I_{n-m} & -\bar{A}_{11}^{-1}\bar{A}_{12} \\ 0 & I_m \end{bmatrix}. \tag{83}$$

The closed loop matrix A_{cl}^c in the new coordinates can be written as

$$A_{cl}^c = \begin{bmatrix} \bar{A}_{11} + \bar{A}_{11}^{-1}\bar{A}_{12}\bar{A}_{21} & 0 \\ \bar{A}_{21} & 0 \end{bmatrix}.$$

The closed loop eigenvalues in this case are clearly seen to be the eigenvalues of the matrix $\bar{A}_{11} + \bar{A}_{11}^{-1}\bar{A}_{12}\bar{A}_{21}$. Hence, stability of the closed loop system implies the stability of the matrix $\bar{A}_{11} + \bar{A}_{11}^{-1}\bar{A}_{12}\bar{A}_{21}$.

Let $(\bar{x}_1)_s$ be the steady state values of the first n elements of $\bar{x}_k$ and $(\bar{x}_2)_s$ be the steady state values of the last m elements of $\bar{x}_k$. Then, the error $\bar{e}_k$ can be partitioned as $(\bar{e}_1)_k = (\bar{x}_1)_k - (\bar{x}_1)_s$ with $(\bar{e}_1)_k \in \mathbb{R}^n$ and $(\bar{e}_2)_k = (\bar{x}_2)_k - (\bar{x}_2)_s$ with $(\bar{e}_2)_k \in \mathbb{R}^m$. It will now be shown that the error dynamics in the uncertain system (64)–(65) converge to the sliding surface $\bar{s}_k$ and that an ideal sliding motion will occur in a finite time. The following theorem will now be stated.

Theorem 4.3
Let the error dynamics of the system (64)–(65) be uniformly ultimately bounded in a finite time $\hat{k}$ and let $\rho_2 = \underset{k>0}{sup}\, \|\hat{D}_k\|$. Then, the error dynamics of the system (64)–(65) converge towards the sliding surface and the sliding surface $\bar{s}_{e_k}$ will be uniformly ultimately bounded with

$$\begin{aligned} \|\bar{s}_{e_{\hat{k}+1}}\| \leqslant{}& \|F_2\bar{A}_{21} + \phi F_2\bar{A}_{21}\bar{A}_{11}^{-1}\|\|(\bar{e}_1)_{\hat{k}}\| \\ &- \left(\|\phi F_2\left[\bar{A}_{21}\bar{A}_{11}^{-1} - I_m\right]\bar{B}_d\| - \|F\bar{C}\bar{B}_d\|\right)\rho_1 \\ &+ \left(\|\phi F_2\Phi^{-1}\bar{\Gamma}_d\| + \|\phi_1\bar{\Gamma}_d\| - \|\bar{\Gamma}_d\|\right)\rho_2 \end{aligned} \tag{84}$$

in a finite time $k > \hat{k} + 1$.

Proof
Consider $\bar{s}_{e_{k+1}}$.

$$\bar{s}_{e_{k+1}} = F\bar{C}\bar{A}_{cl}\bar{e}_k + F\bar{C}\bar{B}(F\bar{C}\bar{A}^{-1}\bar{B})^{-1}\bar{\Gamma}_d\bar{D}_k - \bar{\Gamma}_d\bar{D}_{k+1} + F\bar{C}\bar{B}_d\bar{d}_k \tag{85}$$

Define the matrix ϕ such that $\phi = F_2\bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12}F_2^{-1}$, and let the matrix $\phi_1 = F_2\left(\bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12}\right)F_2^{-1}$. The term $F\bar{C}\bar{A}_{cl}\bar{e}_k$ can be written as

$$F\bar{C}\bar{A}_{cl}\bar{e}_k = F_2\bar{A}_{21}(\bar{e}_1)_k + \phi\bar{s}_{e_k}.$$

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Hence, $\bar{s}_{e_{k+1}}$ can be written as

$$\bar{s}_{e_{k+1}} = F_2\bar{A}_{21}(\bar{e}_1)_k + \phi\bar{s}_{e_k} + \phi_1\bar{\Gamma}_d\bar{D}_k - \bar{\Gamma}_d\bar{D}_{k+1} + F\bar{C}\bar{B}_d\bar{d}_k. \tag{86}$$

Consider the sliding surface dynamics at the $k-1$ instant,

$$\bar{s}_{e_{k-1}} = F\bar{C}\bar{e}_{k-1} - \bar{\Gamma}_d\bar{D}_{k-1} \tag{87}$$

$$= F\bar{C}\left(\bar{A}^{-1}\bar{e}_k - \bar{A}^{-1}\bar{B}\bar{u}_{k-1} - \bar{A}^{-1}\bar{B}_d\bar{d}_{k-1}\right) - \bar{\Gamma}_d\bar{D}_{k-1}. \tag{88}$$

Substituting for $\bar{u}_{k-1}$ in the aforementioned equation gives

$$\bar{s}_{e_{k-1}} = F\bar{C}\left(\bar{A}^{-1}\bar{e}_k - \bar{A}^{-1}\bar{B}\left(\bar{F}\bar{C}\bar{A}^{-1}\bar{B}\right)^{-1}\bar{s}_{e_{k-1}} - \bar{A}^{-1}\bar{B}_d\bar{d}_{k-1}\right) - \bar{\Gamma}_d\bar{D}_{k-1} \tag{89}$$

$$\bar{s}_{e_{k-1}} = F\bar{C}\bar{A}^{-1}\bar{e}_k + F\bar{C}\bar{A}^{-1}\bar{B}\left(\bar{F}\bar{C}\bar{A}^{-1}\bar{B}\right)^{-1}\bar{s}_{e_{k-1}} - F\bar{C}\bar{A}^{-1}\bar{B}_d\bar{d}_{k-1} - \bar{\Gamma}_d\bar{D}_{k-1} \tag{90}$$

$$F\bar{C}\bar{A}^{-1}\bar{e}_k = F\bar{C}\bar{A}^{-1}\bar{B}_d\bar{d}_{k-1} + \bar{\Gamma}_d\bar{D}_{k-1} \tag{91}$$

Letting $\Phi = F_2\left(\bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{21}\right)^{-1}$ and expanding (91) gives

$$-\Phi\bar{A}_{21}\bar{A}_{11}^{-1}(\bar{e}_1)_k + \Phi(\bar{e}_2)_k = \Phi\left[-\bar{A}_{21}\bar{A}_{11}^{-1}\ I_m\right]\bar{B}_d\bar{d}_{k-1} + \bar{\Gamma}_d\bar{D}_{k-1} \tag{92}$$

$$-\bar{A}_{21}\bar{A}_{11}^{-1}(\bar{e}_1)_k + (\bar{e}_2)_k = \left[-\bar{A}_{21}\bar{A}_{11}^{-1}\ I_m\right]\bar{B}_d\bar{d}_{k-1} + \Phi^{-1}\bar{\Gamma}_d\bar{D}_{k-1}. \tag{93}$$

From (93), $\bar{s}_{e_k}$ can be computed as follows:

$$\bar{s}_{e_k} = F_2\bar{A}_{21}\bar{A}_{11}^{-1}(\bar{e}_1)_k - F_2\left[\bar{A}_{21}\bar{A}_{11}^{-1}\ - I_m\right]\bar{B}_d\bar{d}_{k-1} + F_2\Phi^{-1}\bar{\Gamma}_d\bar{D}_{k-1}. \tag{94}$$

Using the aforementioned relation for $\bar{s}_{e_k}$ in (86) and taking norms gives

$$\begin{aligned}\bar{s}_{e_{k+1}} = {} & F_2\bar{A}_{21}(\bar{e}_1)_k + \phi\left(F_2\bar{A}_{21}\bar{A}_{11}^{-1}(\bar{e}_1)_k - F_2\left[\bar{A}_{21}\bar{A}_{11}^{-1}\ - I_m\right]\bar{B}_d.\bar{d}_{k-1} + F_2\Phi^{-1}\bar{\Gamma}_d\bar{D}_{k-1}\right)\\ & + \phi_1\bar{\Gamma}_d\bar{D}_k - \bar{\Gamma}_d\bar{D}_{k+1} + F\bar{C}\bar{B}_d\bar{d}_k\end{aligned} \tag{95}$$

$$\begin{aligned}\|\bar{s}_{e_{k+1}}\| \leqslant {} & \|F_2\bar{A}_{21} + \phi F_2\bar{A}_{21}\bar{A}_{11}^{-1}\|\|(\bar{e}_1)_k\| - \|\phi F_2\left[\bar{A}_{21}\bar{A}_{11}^{-1}\ - I_m\right]\bar{B}_d\|\\ & .\|\bar{d}_{k-1}\| + \|\phi F_2\Phi^{-1}\bar{\Gamma}_d\|\|\bar{D}_{k-1}\| + \|\phi_1\bar{\Gamma}_d\|\|\bar{D}_k\| - \|\bar{\Gamma}_d\|\|\bar{D}_{k+1}\|\\ & + \|F\bar{C}\bar{B}_d\|\|\bar{d}_k\|\end{aligned} \tag{96}$$

$$\begin{aligned}\leqslant {} & \|F_2\bar{A}_{21} + \phi F_2\bar{A}_{21}\bar{A}_{11}^{-1}\|\|(\bar{e}_1)_k\| - \left(\|\phi F_2\left[\bar{A}_{21}\bar{A}_{11}^{-1}\ - I_m\right]\bar{B}_d\|\right.\\ & \left.-\|F\bar{C}\bar{B}_d\|\right)\rho_1 + \left(\|\phi F_2\Phi^{-1}\bar{\Gamma}_d\| + \|\phi_1\bar{\Gamma}_d\| - \|\bar{\Gamma}_d\|\right)\rho_2.\end{aligned} \tag{97}$$

The error $(\bar{e}_1)_k$ on the right-hand side of the aforementioned equation is bounded after a finite time $\hat{k}$. This, in turn, implies that after a finite time $k > \hat{k} + 1$, the sliding surface $\bar{s}_{e_k}$ will be uniformly ultimately bounded, that is,

$$\begin{aligned}\|\bar{s}_{e_{\hat{k}+1}}\| \leqslant {} & \|F_2\bar{A}_{21} + \phi F_2\bar{A}_{21}\bar{A}_{11}^{-1}\|\|(\bar{e}_1)_{\hat{k}}\| - \left(\|\phi F_2\left[\bar{A}_{21}\bar{A}_{11}^{-1}\ - I_m\right]\bar{B}_d\|\right.\\ & \left.-\|F\bar{C}\bar{B}_d\|\right)\rho_1 + \left(\|\phi F_2\Phi^{-1}\bar{\Gamma}_d\| + \|\phi_1\bar{\Gamma}_d\| - \|\bar{\Gamma}_d\|\right)\rho_2.\end{aligned} \tag{98}$$

□

Remark 6

For the regulation case with $r_k = 0$, consider the system (1)–(2) in the coordinate system (51) with the control law

$$\bar{u}_k = -(F\bar{C}\bar{A}^{-1}\bar{B})^{-1}\bar{s}_k, \tag{99}$$

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and where the sliding surface $\bar{s}_k$ is defined as follows:

$$s_k = F\tilde{y}_k + \Gamma_u U_k \tag{100}$$
$$= F\tilde{C}x_k - \Gamma_d D_k. \tag{101}$$

From the aforementioned, the following closed loop system can then be formed

$$\bar{x}_{k+1} = \bar{A}_{cl}\bar{x}_k + \bar{B}_D\bar{w}_k \tag{102}$$
$$\bar{Y}_k = \bar{C}\bar{x}_k - \bar{\Gamma}_1 w_k \tag{103}$$
$$\bar{z}_k = \begin{bmatrix} \bar{Y}_k \\ \bar{s}_k \end{bmatrix}, \tag{104}$$

where $\bar{z}_k$ is a performance output. An analysis similar to the analysis performed for the tracking case can then be performed for the closed loop system (102) along with the sliding surface (101) to show that the states will be uniformly ultimately bounded around the sliding surface. For the regulation case, note that the condition $\text{rank}(C_p B_p) = m$ is not required to be satisfied, and the algorithm proposed in this paper can be useful for designing output-based discrete time sliding mode control for systems including overactuated systems.

The following section will now discuss the design of a tracking control for a benchmark aircraft problem taken from the literature.

5. F-14 LATERAL DYNAMICS

The linearized aircraft model used in this example is obtained at an angle of attack of 10.5 degs and an airspeed of 137 knots [15, 16]. The pilot can command the lateral directional response of the aircraft with the lateral stick and the rudder pedals. The control inputs for the aircraft are the differential stabilizer deflection (δ_{stab}, deg) and the rudder deflection (δ_{rud}, deg). The measured outputs are the yaw rate (r, deg/sec), roll rate (p, deg/sec), and the lateral acceleration (y_{ac}, g's). There is also available one calculated output that is the side slip angle (β, deg) . The states for the nominal lateral dynamics model of the F-14 are the lateral velocity (v), yaw rate (r), roll rate (p), and the roll angle (ϕ, deg). The linearized plant state space model is described by the following continuous time state space model

$$\dot{x} = A_c x(t) + B_c u(t) \tag{105}$$
$$y(t) = C_c x(t) + D_c u(t), \tag{106}$$

where

$$A_c = \begin{bmatrix} -0.116 & -227.3 & 43.02 & 31.63 \\ .00265 & -0.259 & -0.1445 & 0 \\ -0.02114 & 0.6703 & -1.365 & 0 \\ 0 & 0.1853 & 1 & 0 \end{bmatrix} B_c = \begin{bmatrix} 0.0622 & 0.1013 \\ -0.005252 & -0.01121 \\ -0.04666 & 0.003644 \\ 0 & 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 0.2469 & 0 & 0 & 0 \\ 0 & 0 & 57.2958 & 0 \\ 0 & 57.2958 & 0 & 0 \\ -0.0028 & -0.0079 & 0.0511 & 0 \end{bmatrix} D_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.0029 & 0.0023 \end{bmatrix}$$

with state vector $x^T = [v\ r\ p\ \phi]$, with inputs $u^T = [\delta_{\text{dstab}}\ \delta_{\text{rud}}]$ and outputs $y^T = [\beta\ p\ r\ y_{\text{ac}}]$.

The aircraft dynamics also includes models of the differential stabilizers (A_S) and the rudders (A_R). The differential stabilizer model A_S is modeled as $A_s = \text{diag}\left(\frac{25s}{s+25}, \frac{25}{s+25}\right)$ and the rudder actuator model A_R is modeled as $A_R = A_S$. The actuator models are included in the overall

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aircraft system representation considered for the control design, and only the second output from each actuator model will be considered for the same. The differential stabilizers and the rudder actuator models have saturations on the deflection and deflection rate limits. The deflection and deflection rate limits for these actuators are ± 20 deg and ± 90 deg/s, respectively, for the differential stabilizers and ± 30 deg and ± 125 deg/sec, respectively, for the rudders.

The aircraft model discussed earlier is an approximate model. The unmodeled dynamics is accounted for by introducing a relative term or multiplicative uncertainty $W_{in}\Delta_G$ at the plant input. The error dynamics Δ_G is stable and chosen such that it has a gain less than one across all frequencies. The weighting function W_{in} reflects the frequency ranges in which the model is more or less accurate. The weighting function W_{in} is chosen as high pass because there are more modeling errors at high frequencies and is selected as $w_1 = \frac{2(s+4)}{s+160}$ and $w_2 = \frac{2(s+20)}{s+200}$. Hence, the true airplane model would be

$$G = (I + W_{in}\Delta_G)\bar{G},$$

where $\bar{G}$ is defined as follows:

$$\bar{G} = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{107}$$

The nominal model can now be combined with the actuator models A_S and A_R and the uncertain model $W_{in}\Delta_G$ in to a single uncertain system representation. The uncertain model thus obtained describes a family of plants. Using the nominal model of the uncertain plant obtained from MATLAB and then discretizing the model at 60 Hz [15] gives a linear time invariant uncertain discrete time model as given as follows:

$$A_d = \begin{bmatrix} 0.9979 & -3.7711 & 0.7171 & 0.5267 & 0.0006 & 0.0013 \\ 0.0000 & 0.9956 & -0.0024 & 0.0000 & -0.0001 & -0.0001 \\ -0.0003 & 0.0117 & 0.9774 & -0.0001 & -0.0005 & 0.0000 \\ -0.0000 & 0.0032 & 0.0165 & 1.0000 & -0.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & 0.6592 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6592 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0.0002 & 0.0004 \\ -0.0000 & -0.0000 \\ -0.0001 & 0.0000 \\ -0.0000 & 0.0000 \\ 0.4362 & 0 \\ 0 & 0.4362 \end{bmatrix}$$

$$C_d = \begin{bmatrix} 0.2469 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 57.2958 & 0 & 0 & 0 \\ 0 & 57.2958 & 0 & 0 & 0 & 0 \\ -0.0028 - 0.0079 & 0.0511 & 0 & 0.0023 & 0.0018 & \end{bmatrix} D_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The aforementioned uncertain model will now be used for control system design. The aim here is to design a tracking control for tracking the sideslip of the aircraft when the aircraft loses the yaw rate measurement r. The original controller design [15] does not stabilize the system when the yaw rate measurement is lost. Hence, the basic design goal here in this paper is to design an extended output feedback-based control that will stabilize the aircraft and will provide a tracking control for the plant. The control design given in this paper will not consider the impact of gusts on the aircraft

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or sensor noise in the design. Hence, the matrix B_{p_d} in (1) is assumed as zero. The uncertain plant is augmented with an integrator as given in (7). The resulting plant matrices are the following:

$$A = \begin{bmatrix} 1.0000 & -0.2469 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9979 & -3.7711 & 0.7171 & 0.5267 & 0.0006 & 0.0013 \\ 0 & 0.0000 & 0.9956 & -0.0024 & 0.0000 & -0.0001 & -0.0001 \\ 0 & -0.0003 & 0.0117 & 0.9774 & -0.0001 & -0.0005 & 0.0000 \\ 0 & -0.0000 & 0.0032 & 0.0165 & 1.0000 & -0.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & 0 & 0.6592 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6592 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0.0002 & 0.0004 \\ -0.0000 & -0.0000 \\ -0.0001 & 0.0000 \\ -0.0000 & 0.0000 \\ 0.4362 & 0 \\ 0 & 0.4362 \end{bmatrix} \quad B_r = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2469 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57.2958 & 0 & 0 & 0 \\ 0 & 0 & 57.2958 & 0 & 0 & 0 & 0 \\ 0 & -0.0028 & -0.0079 & 0.0511 & 0 & 0.0023 & 0.0018 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

There are no invariant zeros in the original plant triple $(A_d,\ B_d,\ C_d)$, and hence, according to Lemma 2.1, the augmented triple $(A,\ B,\ C)$ does not possess any invariant zeros. The extended outputs that are used for the reconfigurable control design are obtained by taking two past measurements of the roll rate p. Hence, the outputs used for the reconfigurable control design includes three current outputs $\beta,\ p,\ \phi$, and two past measurements of the roll rate p. Hence, the augmented output matrix $\tilde{C}$ will be the following:

$$\tilde{C} = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2469 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57.2958 & 0 & 0 & 0 \\ 0 & 0.0204 & -0.6102 & 58.6061 & -0.0054 & 0.0437 & -0.0034 \\ 0 & 0.0414 & -1.1593 & 59.9302 & -0.0216 & 0.1108 & -0.0088 \\ 0 & -0.0028 & -0.0079 & 0.0511 & 0 & 0.0023 & 0.0018 \end{bmatrix}.$$

It can be checked that the triple $A,\ B,\ \tilde{C}$ does not have any invariant zeros. The transformation matrix T used to transform the system triple $(A,\ B,\ \tilde{C})$ into the form in (32) is as follows:

$$T = \begin{bmatrix} -0.0227 & -0.9472 & -0.2563 & 0.1914 & -0.0000 & -0.0000 \\ -0.9936 & 0.0313 & -0.0072 & 0.0276 & -0.1046 & -0.0010 \\ 0.0290 & 0.1805 & 0.0492 & 0.9627 & 0.0274 & 0.1914 \\ -0.0121 & -0.2426 & 0.9342 & 0.0492 & -0.0069 & -0.2563 \\ 0.0076 & 0.1019 & -0.2426 & 0.1805 & 0.0323 & -0.9471 \\ 0.1058 & 0.0067 & -0.0123 & 0.0292 & -0.9936 & -0.0236 \end{bmatrix},$$

and the subsystem $(\tilde{A}_{11}, \hat{A}_{122}, \tilde{C}_f)$ used to design the sliding surface is

$$\tilde{A}_{11} = \begin{bmatrix} 2.3801 & -16.5566 & 61.4460 & -160.0460 & 87.7697 \\ 0.0970 & -0.2404 & 3.7367 & -9.8904 & 5.0990 \\ -0.0026 & -0.9068 & 0.9582 & 0.1911 & -0.0585 \\ 0.0008 & -0.2175 & 0.5633 & -0.3058 & 1.0129 \\ -0.0001 & 0.2653 & 0.8268 & -1.9383 & 2.5273 \end{bmatrix}$$

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$$\hat{A}_{122} = \begin{bmatrix} 3.8473 & 4.3088 \\ 0.2428 & 0.0103 \\ -0.1000 & 0.0128 \\ 0.0164 & 0.2646 \\ 0.1026 & 0.4241 \end{bmatrix} \quad \tilde{C}_f = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The state feedback gain K_o is obtained by placing the poles at [0.95 0.983 0.964 0.975 0.97]. The triple $(\tilde{A}_{11}, \hat{A}_{122}, \tilde{C}_f)$ is used to compute the gain K using the algorithm given in [19] and is obtained as follows:

$$K = \begin{bmatrix} -0.3406 & -5.5050 & 14.0158 & -8.5276 \\ 7.8789 & 81.0530 & -206.7674 & 125.5446 \end{bmatrix},$$

with $\lambda(\tilde{A}_{11} - \hat{A}_{122} K \tilde{C}_f)$ at $[0.9337 + 0.0334i \;\; 0.9337 - 0.0334i \;\; 0.9721 + 0.0048i \;\; 0.9721 - 0.0048i \;\; 0.9838]$. The output feedback gain F is computed as

$$F = \begin{bmatrix} 0.0029 & 0.0657 & 8.5502 & -14.0209 & 5.5351 & -0.4997 \\ -0.0804 & 0.3340 & -125.3461 & 206.4925 & -82.0861 & -7.6029 \end{bmatrix},$$

and the matrix Γ_u as

$$\Gamma_u = \begin{bmatrix} 0.0695 & -0.0057 & 0.0598 & -0.0047 \\ -1.0460 & 0.0856 & -0.8869 & 0.0690 \end{bmatrix}.$$

With the aforementioned control gain, the closed loop eigenvalues are at $[0.9474+0.0625i \;\; 0.9474-0.0625i \;\; 0.9897 \;\; 0.9773+0.0044i \;\; 0.9773-0.0044i \;\; 0.0000 \;\; 0.0000]$. The reference input is a $\pm\, 1\, in$ lateral pedal input, and the control law (56) is applied to both the nominal F14 model and the uncertain model. The response of the aircraft for the lateral pedal reference input is shown in Figures 1–3 for both the nominal model and the uncertain model. In both responses, the actuation signals are within the saturation limit. It can be seen that the responses of both the nominal plant and the uncertain model are nearly identical. Considering the fact that the original state feedback design in [15] does not stabilize the plant in the absence of the yaw rate measurement, the control law (56) is shown to provide stabilization and sideslip β tracking in the absence of the same sensor loss. The sideslip β follows the pedal command as can be seen from the responses. Also, easily noticeable from the simulation results is the strong coupling between the roll and the yaw dynamics. One of the main criteria in the handling qualities requirement in the original design was the complete decoupling of the roll rate and the side slip angle that is not recoverable here with the aforementioned control law, and hence, the aforementioned control design comes with the cost of increased roll rate response

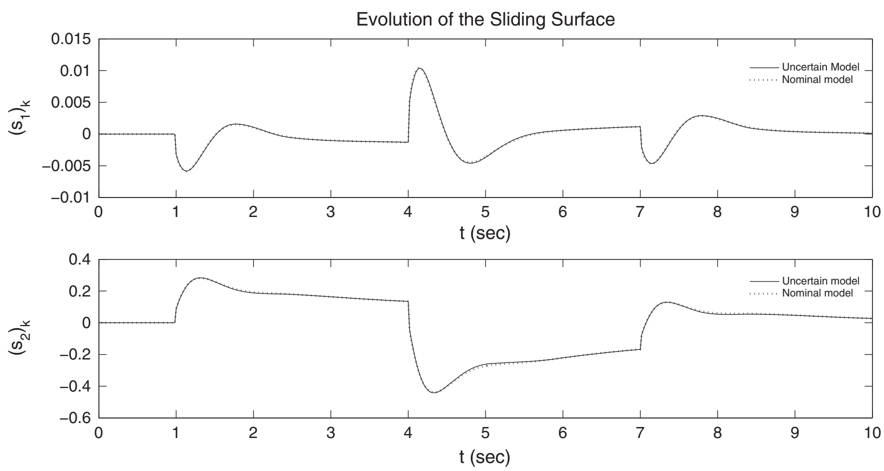

Figure 1. Sliding surface dynamics.

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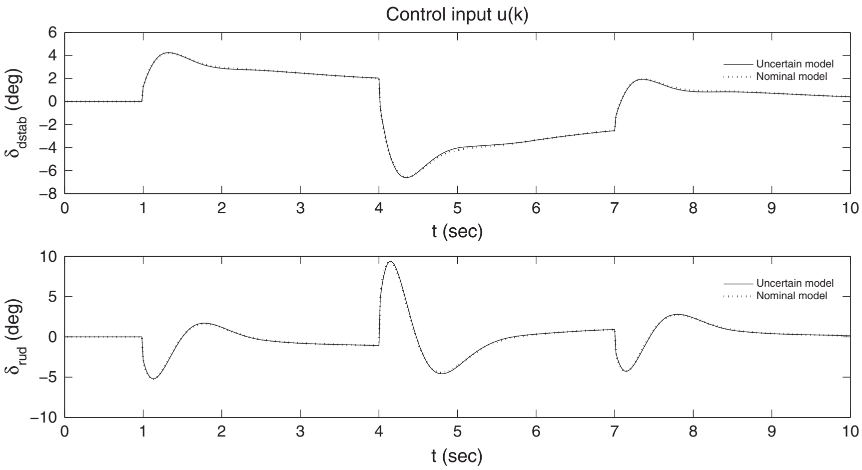

Figure 2. The control inputs applied to the plant.

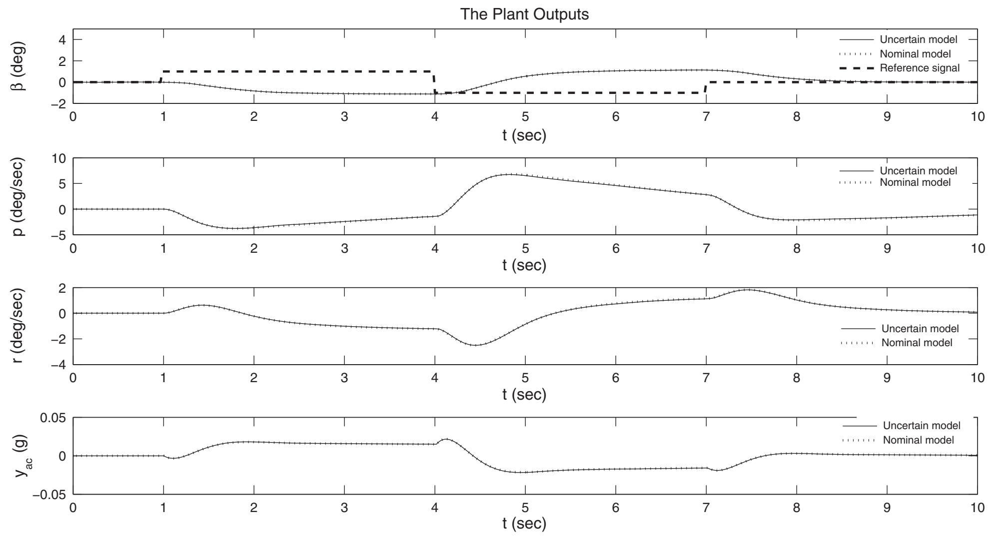

Figure 3. The outputs of the aircraft in the presence of uncertainty.

and the loss of decoupling between roll rate and the side slip angle β when a lateral pedal input is applied. The aforementioned control design, hence, has both its advantages and disadvantages: stability and some performance is recovered under a sensor failure, but there is a loss of decoupling between the roll rate and the side slip angle.

6. CONCLUSION

An output feedback-based sliding mode control design for discrete time systems that incorporates integral action for tracking control has been developed in this paper. It has been shown that discrete time controllers can be realized via the extended outputs for nonsquare systems with uncertainties. The conditions for the existence of a sliding mode have been given. A procedure for synthesizing a control law has also been given. The control law has been chosen such that the norm of a particular closed loop transfer function is minimized. The efficacy of the control law has been shown by applying the control law to a benchmark aircraft problem taken from the literature.

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