Derivative Based Control For LTV System With Unknown Parameters

Sonia MAALEJ, Alexandre KRUSZEWSKI, Lotfi BELKOURA

***Abstract*— This paper deals with the robust stabilization of a class of linear time varying systems. Instead of using a state observer or having dynamic structure, the controller is based on output derivative estimation. This allows the stabilization of linear time varying systems with very large parameter variation and a small number of controller parameters. The proof of stability is based on the polytopic representation of the closed loop and Lyapunov conditions. The result is proposed in a Linear Matrix Inequality (LMI) form. The validity of this approach is illustrated by a second order system case of study.**

***Index Terms*— Linear Time Varying systems, Robust control, Uncertainties, Linear Matrix Inequalities, Polytopic systems.**

I. Introduction

Since many years, state feedbacks and dynamic output feedbacks [1] are largely used to guarantee stability and performances. Despite the fact that dynamic output feedbacks are more robust than observer based controllers [2] [3], they still need a model of the plant without too much uncertainties nor time varying parameters with a large variation set. Getting a precise model is time consuming and the controller may become complex if the time varying parameters are the sources of instability. Considering more universal controllers with simple design procedure seems to be important in that context especially if the controlled plant is just an actuator or an auxiliary subsystem.

Recently, some works have addressed issues related to controllers with an another point of view. For exemple, Cedric et al. [4] introduced a *model free* controller. The latter is signal based and thus don't require many informations about the system. To implement this controller with the less dependence on a good model, the output derivatives are used by the controller.

In the literature, there exist many techniques allowing the derivative estimation of a signal. The simplest one is to use a filtered differentiator i.e. any stable transfer function with a pure differentiator for the numerator. Other possibilities consist in the use of algebraic differentiators [5] ,[4], unknown input Luenberger observer or sliding mode observers [6], [7], [8], [9].

In this paper, a controller inspired by [4] based on the simplest differentiator is presented. This choice allows to study the stability of the closed loop for a class of systems. This study is performed through the Lyapunov theory and thanks to a polytopic transformation, the stability conditions will be presented as a set of LMI (Linear Matrix Inequalities). In a first part, the class of considered systems is presented, then, the controller and the used differentiation technique are given. The second part provides a polytopic representation of the closed loop and the global stability conditions. The third one is a study case for second order system providing more specific conditions. Some numerical examples are then presented showing the effectiveness of the controller on a set of system with a large parameter variations. The last part gives some conclusions and perspectives.

II. Problem formulation

A. The class of system

In this paper, we consider the following class of single input single output linear system with time varying parameters :

$$y^{(n)}(t) = -a_0(t)y(t) - \ldots - a_{n-1}(t)y^{(n-1)}(t) + \alpha u(t). \quad (1)$$

where $y(t) \in \mathbb{R}$ is the output, $u(t) \in \mathbb{R}$ is the control input and a_i are scalar time varying parameters.

The aim of many researches is to provide the most parameter independent controller ensuring the stabilization of this system.

To achieve the proposed goal, the system will be viewed as the following equation from the controller point of view :

$$y^{(n)}(t) = f(t) + \hat{\alpha}u(t) \quad (2)$$

that means that only the order and an approximation of the parameter α are given. In the formulation of the system equation given in (2), $f(t)$ is the function which contains all the dynamic information, i.e. the global dynamic of the system and the possible disturbances. The function $f(t)$ MUST be control independent.

S. Maalej and L. Belkoura are with the Université de Lille 1, LAGIS (Laboratoire d'Automatique, Génie Informatique et Signal, CNRS UMR 8219), 59651 Villeneuve d'Ascq, France sonia.maalej@ec-lille.fr

S. Maalej and L. Belkoura are with the Non-A, INRIA Lille-Nord Europe, France

A. Kruszewski is with the Ecole Centrale de Lille, LAGIS (Laboratoire d'Automatique, Génie Informatique et Signal, CNRS UMR 8219), 59651 Villeneuve d'Ascq, France alexandre.kruszewski@ec-lille.fr

This work was supported by the INTERREG IV A 2 Mers Seas Zeen Cross-border Cooperation Program 2007-2013 under SYSIASS 6-20 project.

B. The controller

Since the only information on the system are given by Equation (2), the idea of the controller structure is to nullify the system dynamic $f(t)$ and then replace it with the ideal dynamic for the closed loop. Considering this idea, we get the following controller :

$$u(t) = -\frac{1}{\hat{\alpha}}(\hat{f}(t) + K\hat{Y}(t) - k_0 r(t)). \quad (3)$$

where $\hat{f}(t)$ is the estimate of $f(t)$, $\hat{\alpha}$ is an approximation of α, $r(t)$ is the reference, $\hat{Y}(t) = [z_0(t) \quad \dots \quad z_{n-1}(t)]^T$ is a vector composed with $z_i(t)$ the estimation of $y^{(i)}(t)$ and $K = [k_0 \quad \dots \quad k_{n-1}]$ is a vector with the coefficients of the ideal closed loop dynamic given by the specifications.

C. Derivative estimation

In this part, the problem of estimation of $f(t)$ and the successive derivatives of $y(t)$ are considered. In the literature, many approaches were proposed : one can consider sliding mode estimation, [6], [7], [8], [9], Luenberger observer based estimation and algebraic estimation [5]. In order to get a simple solution, we consider a simple filtered derivative approach :

$$\begin{cases} \frac{z_0(s)}{y(s)} = \frac{1}{\tau s+1} \\ \frac{z_i(s)}{y(s)} = E(s)^i = \left(\frac{s}{\tau s+1}\right)^{i+1} \quad \forall i = 1 \dots n \end{cases} \quad (4)$$

This estimator is causal and ensures a good estimation if τ is sufficiently smaller than the fastest dynamic of the system. It provides the successive estimations $z_i(t)$ of $y^{(i)}$.

The estimation of $f(t)$ is made by inverting the dynamic Equation (2) but instead of using the value of $u(t)$, a filtered version of it must be used. This is mandatory to avoid the algebraic loop between $u(t)$ and $\hat{f}(t)$.

$$\hat{f}(t) = z_n(t) - \hat{\alpha}\hat{u}(t) \quad (5)$$

with $\frac{\hat{u}(s)}{u(s)} = \frac{1}{\tau s+1}$ and $z_n(t)$ being the estimation of $y^{(n)}(t)$.

III. Main result

The stability of the closed loop needs to be proven. Since it is impossible to stabilize all the models in the form of (1), the stability for a set of variation of the parameters is studied. The conditions used come from the polytopic framework.

A. Closed loop state space representation

The state space representation of the different dynamic equation of the last section are given here. The model in the state space form is given by choosing the state vector as $x_m(t) = [y(t) \quad \dots \quad y^{(n-1)}(t)]^T$:

$$\begin{cases} \dot{x}_m(t) = A_m(t)x_m(t) + B_m u(t) \\ y(t) = C_m x_m(t) \end{cases} \quad (6)$$

Where :

$$A_m(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \dots & \dots & \dots & 1 \\ -a_0(t) & -a_1(t) & \dots & \dots & -a_{n-1}(t) \end{pmatrix},$$

$$B_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \alpha \end{pmatrix} \text{ and } C_m = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \end{pmatrix},$$

The estimator can be represented as it follows where $\dot{x}_e(t) = [z_1(t) \quad \dots \quad z_n(t) \quad \hat{u}(t)]^T$ (see Proof 1) :

$$\begin{cases} \dot{x}_e(t) = A_e x_e(t) + B_{ey} y(t) + B_{eu} u(t) \\ \hat{Y}(t) = C_{eyx} x_e(t) + C_{eyy} y(t) \\ \hat{u}(t) = C_{eu} x_e(t) \end{cases} \quad (7)$$

With :

$$A_e = \begin{pmatrix} -\frac{1}{\tau} & 0 & \dots & \dots & \dots & 0 & 0 \\ -\frac{1}{\tau^2} & -\frac{1}{\tau} & 0 & \dots & \dots & 0 & 0 \\ -\frac{1}{\tau^3} & -\frac{1}{\tau^2} & -\frac{1}{\tau} & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -\frac{1}{\tau^{n-1}} & -\frac{1}{\tau^{n-2}} & \dots & \dots & -\frac{1}{\tau} & 0 & 0 \\ -\frac{1}{\tau^n} & -\frac{1}{\tau^{n-1}} & \dots & \dots & \dots & -\frac{1}{\tau} & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 & -\frac{1}{\tau} \end{pmatrix},$$

$$B_{ey}^T = \begin{pmatrix} \frac{1}{\tau} & \frac{1}{\tau^2} & \dots & \frac{1}{\tau^n} & 0 \end{pmatrix}, \; B_{eu}^T = \begin{pmatrix} 0_{(1\times n)} & \frac{1}{\tau} \end{pmatrix},$$

$$C_{eyx} = \begin{pmatrix} 1 & 0_{(1\times n)} \\ A_{e_{((1:n-1)\times(1:n))}} & 0 \end{pmatrix},$$

$$C_{eyy}^T = \begin{pmatrix} 0 & B_{ey_{(1:n-1)}} \end{pmatrix} \text{ and } C_{eu} = \begin{pmatrix} 0_{(1\times n)} & 1 \end{pmatrix}.$$

Then, the estimation of the dynamic $f(t)$ is :

$$\hat{f}(t) = C_{fx} x_e(t) + C_{fy} y(t),$$

where :

$$C_{fx} = \begin{pmatrix} -\frac{1}{\tau^n} & -\frac{1}{\tau^{n-1}} & \dots & -\frac{1}{\tau} & -\hat{\alpha} \end{pmatrix} \text{ and } C_{fy} = \frac{1}{\tau^n}.$$

Finally the closed loop of the system and its controller are :

$$\begin{cases} \dot{x}_b(t) = \tilde{A}(t)x_b(t) + \tilde{B}r(t) \\ y_b(t) = \tilde{C}x_b(t) \end{cases}, \quad (8)$$

with the closed system state $x_b(t) = [x_m(t) \; x_e(t)]^T$,

$$\tilde{A}(t) = \begin{pmatrix} A_m(t) + \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \tilde{B} = \frac{k_0}{\hat{\alpha}} \begin{pmatrix} B_m \\ B_{eu} \end{pmatrix}$$

$$\text{and } \tilde{C} = \begin{pmatrix} 0_{(1\times 2n-1)} & 1 & 0 \end{pmatrix}.$$

where :
$\tilde{A}_{11} = -\frac{1}{\hat{\alpha}} B_m (\frac{1}{\tau^n} + K C_{eyy}) C_m$
$\tilde{A}_{12} = -\frac{1}{\hat{\alpha}} B_m (C_{fx} + K C_{eyx})$
$\tilde{A}_{21} = B_{ey} C_m - \frac{1}{\hat{\alpha}} B_{eu} (\frac{1}{\tau^n} + K C_{eyy}) C_m$
$\tilde{A}_{22} = A_e - \frac{1}{\hat{\alpha}} B_{eu} (C_{fx} + K C_{eyx})$

B. Polytopic representation and stability analysis

There exist several ways to derive stability conditions. Here we chose to represent the closed loop system given by Equation (8) as a polytopic system [10]. Then the stability conditions are given in the form of a set of LMI to satisfy. Since the closed loop matrices are assumed to be bounded, they can be described/over-approximated by a convex combination of matrices :

$$\mathcal{A} = \{\tilde{A}(t) : \tilde{A}(t) = \sum_{i=0}^{2^n} \mu_i(t) \tilde{A}_i;\ \mu_i(t) \in \Delta_\mu\}; \quad (9)$$

$$\Delta_\mu = \{\mu_i(t) \in \mathbb{R}^{2^n} :\ \sum_{i=0}^{2^n} \mu_i(t) = 1;\ 0 \leq \mu_i(t) \leq 1\}.$$

Where n is the number of time varying parameters. The vertices $\tilde{A}_i$ of this convex form can be derived by using the using the non-linear sector approach. The stability theorem of this class of system is given by :

***Theorem** 1:* If it exist a matrix $P = P^T > 0$ such that :

$$\tilde{A}_i^T P + P \tilde{A}_i < 0, \quad \forall i = 1, \ldots, 2^n \quad (10)$$

then, the closed loop 8 is asymptotically stable.

The proof is obvious considering the results of [10].

IV. Study case : 2^{nd} order LTV system

The stability of the closed loop for a given system was proven in the last section. Since this study aims to allow the stabilization of the largest set of system with the same controller (with the less information possible), the second order case is considered :

$$A_m(t) = \begin{pmatrix} 0 & 1 \\ -a_0(t) & -a_1(t) \end{pmatrix},$$
$$B_m = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \text{ and } C_m = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

***Remark** 1:* For this study case, the closed loop dynamic is chosen to be given by $K = [1\ 2]$ (i.e. a closed loop differential equation given by $\ddot{y}(t) = r(t) - y(t) - 2\dot{y}(t)$) and the value of τ is given by $\tau = 0.01$.

A. Linear Time Invariant case analysis

The first result considers the Linear Time Invariant (LTI) case i.e. the parameters $a_0(t) = a_0$ and $a_1(t) = a_1$ of the system dynamic are constant and unknown. The Fig. 1 shows the stability region according to the values of a_0, $-a_1$ and the value of $\frac{\alpha}{\hat{\alpha}}$.

We note from this figure (Fig. 1) that the controller is robust and guarantees the stability even for a large variation of the parameters.

The Fig. 2 shows the state trajectory for the case $[a_0,\ a_1] = [20,\ -40]$ (i.e. the system is unstable). From this figure, we note that this estimator is perferment. In fact, the system output converge to the one imposed by the model function. Also, Fig 3 and Fig 4 show that the model and the control outputs follow rapidly their estimate values.

B. Linear Time Varying case analysis

This second result considers the LTV case. We consider that $a_0(t) \in [\underline{a}_0\ \overline{a}_0]$ and $a_1(t) \in [\underline{a}_1\ \overline{a}_1]$. This implies that the vertices of the polytopic model are given by :

$$A_{m_1}(t) = \begin{pmatrix} 0 & 1 \\ -\underline{a}_0 & -\underline{a}_1 \end{pmatrix},\ A_{m_2}(t) = \begin{pmatrix} 0 & 1 \\ -\underline{a}_0 & -\overline{a}_1 \end{pmatrix}$$

$$A_{m_3}(t) = \begin{pmatrix} 0 & 1 \\ -\overline{a}_0 & -\overline{a}_1 \end{pmatrix},\ A_{m_4}(t) = \begin{pmatrix} 0 & 1 \\ -\overline{a}_0 & -\underline{a}_1 \end{pmatrix}$$

***Remark** 2:* For this case, the parameters bonded are chosen such that $\overline{a}_{0,1} = a_{0,1}$ and $\underline{a}_{0,1} = -a_{0,1}$.

The Fig. 5 shows the maximum value of a_1 allowed for a given value of a_0. This set decreases with $\frac{\alpha}{\hat{\alpha}}$ making the approximation of the parameter α an important point.

Then, Fig. 6 presents the state trajectory for the case $a_0 = 20\,sin(t)$ and $a_1 = -40\,sin(t)$. Fig. 6 shows that the system output converge to the one imposed by the model function.

Fig 7 and Fig 8 show the model and control output evolution when noises are added to the system output.

***Remark** 3:* In both cases (LTI and LTV case), the stability region of the system is directly linked to the value of τ. These sets grow when τ decreases. In fact, if the signals vary quicker than the time constant τ, the estimation error of the system output derivatives as well as the open loop dynamic $f(t)$ will be increased.

V. Conclusions

This paper provides a robust controller based on output derivative estimation. Stability conditions are provided for the case where the open loop system has bounded time varying parameters. A study case shows that this controller can stabilize a large set of LTI/LTV second order system if an approximation of some parameters or their bounds on the parameters are available. Our future work will be devoted to deriving conditions without solving any LMI problems and studying system performance.

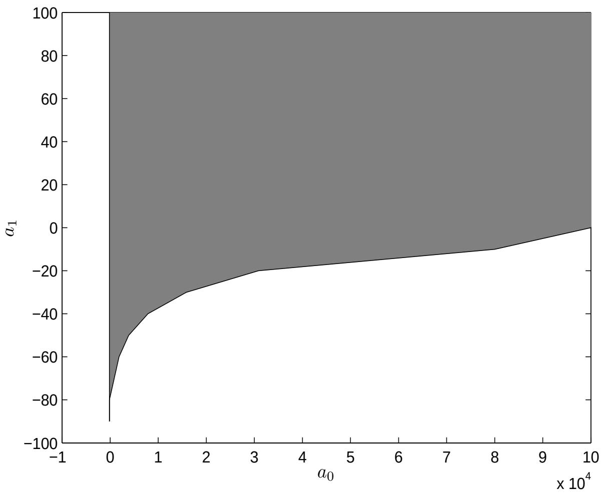

Fig. 1. Stability region of the LTI case (zoom in)

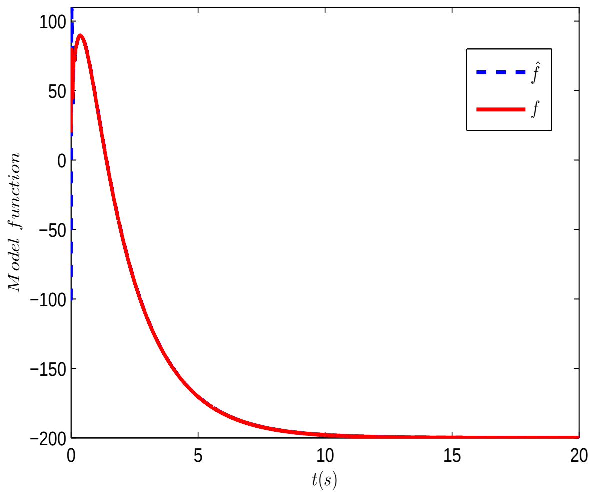

Fig. 3. Evolution of the model function and its estimate for the Linear case with $r(t) = 10$

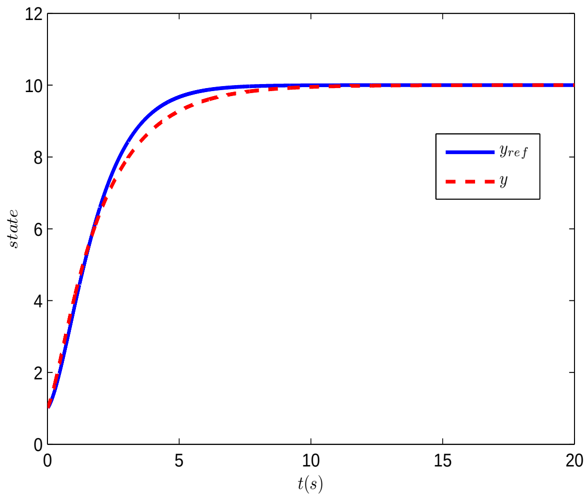

Fig. 2. State evolution for the LTI case with $r(t) = 10$

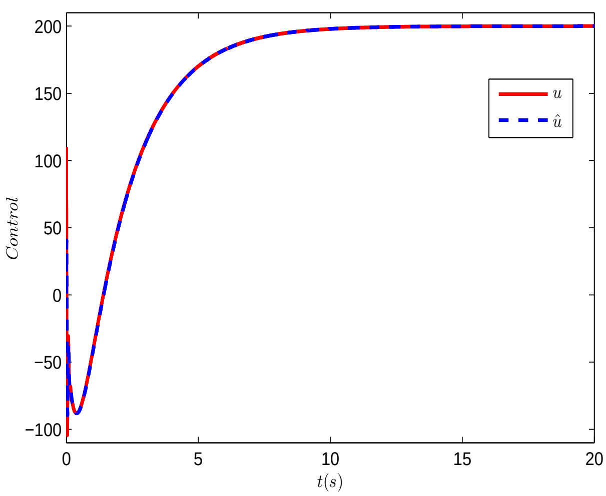

Fig. 4. Evolution of the control function and its estimate for the Linear case with $r(t) = 10$

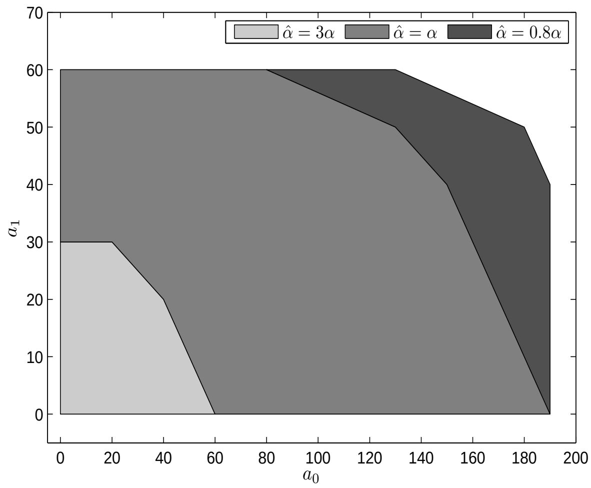

Fig. 5. Stability region for different values of $\frac{\alpha}{\hat{\alpha}}$ of the LTV system

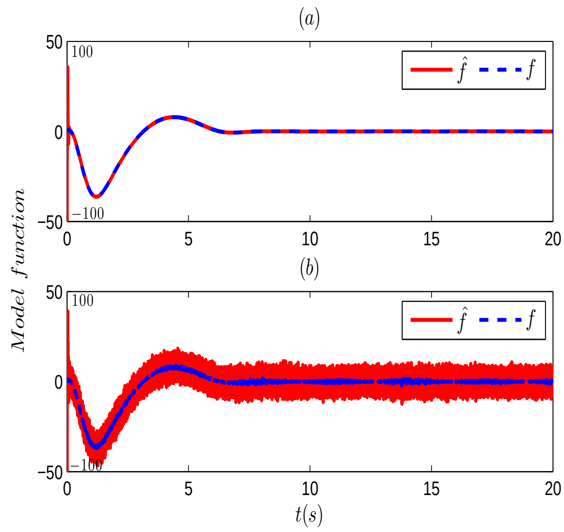

Fig. 7. Evolution of the model function and its estimate for the LTV system with $r(t) = 0$: (a)-without noise injection and (b)-with noise injection

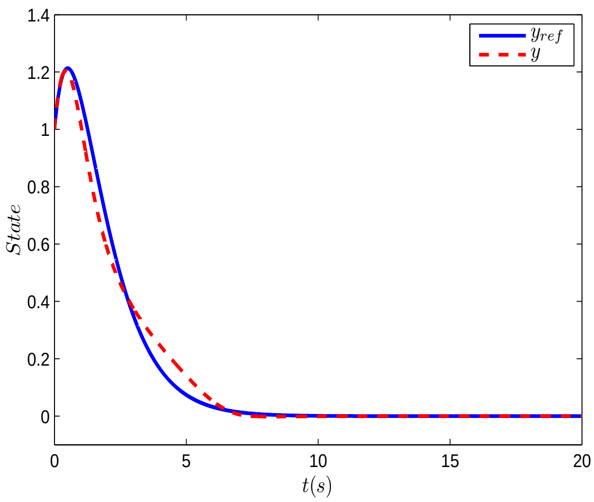

Fig. 6. State evolution for the LTV case with $r(t) = 0$

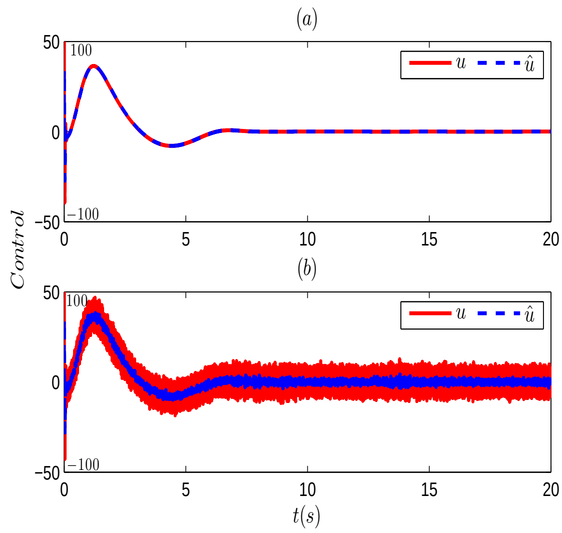

Fig. 8. Evolution of the control function and its estimate for LTV sustem with $r(t) = 0$: (a)-without noise injection and (b)-with noise injection

VI. Appendix

***Proof** 1:*
We distinguish tow cases :

- For $n = 1$:
  For this case, the estimator has $y(s)$ as input and $z_1(s)$ as output. From (4), the transfer function of this estimator is given by :
  $$\frac{z_1(s)}{y(s)} = \frac{s}{\tau s + 1}.$$
  This implies :
  $$y(s)s = \tau s z_1(s) + z_1(s).$$
  Then, we note $x_{e_i}(t) = \int z_i(t)dt$ and by integrating one time the result we get :
  $$z_1(t) = \frac{1}{\tau}(y(t) - x_{e_1}(t)) \quad (11)$$
- For $n > 1$:
  In this case, the transfer function is given by :
  $$\frac{z_n(s)}{z_{n-1}(s)} = \frac{s}{\tau s + 1}. \quad (12)$$
  We improve by recurrency that for $k = 2 \dots n-1$:
  $$z_n(t) = \frac{1}{\tau^n} y(t) - \frac{1}{\tau^n} x_{e_1}(t) - \dots - \frac{1}{\tau^{n-k}} x_{e_k}(t). \quad (13)$$
  First of all, let us verify if (13) is true for $n = 2$. From (4) we have :
  $$\frac{z_2(s)}{z_1(s)} = \frac{s}{\tau s + 1}.$$
  $$s z_1(s) = \tau s z_2(s) + z_2(s).$$
  Or:
  $$z_2(t) = \frac{1}{\tau}(z_1(t) - x_{e_2}(t)).$$
  Using (11):
  $$\begin{aligned} z_2(t) &= \frac{1}{\tau}(-\frac{1}{\tau}x_{e_1}(t) + \frac{1}{\tau}y(t) - x_{e_2}(t)) \\ &= \frac{1}{\tau^2}y(t) - \frac{1}{\tau^2}x_{e_1}(t) - \frac{1}{\tau}x_{e_2}(t)(t). \end{aligned}$$
  This proves that (13) is true for $n = 2$. Now let us check if equation (13) is true for $n = k$, it is olso for $n = k + 1$.
  $$\frac{z_{n+1}(s)}{z_n(s)} = \frac{s}{\tau s + 1}.$$
  Thus:
  $$s z_n(s) = \tau s z_{n+1}(s) + z_{n+1}(s).$$
  Then, after one integration:
  $$z_{n+1}(t) = \frac{1}{\tau}(z_n(t) - x_{e_{n+1}}(t))$$
  Then, from (13):
  $$\begin{aligned} z_{n+1}(t) &= \frac{1}{\tau}(\frac{1}{\tau^n}y(t) - \frac{1}{\tau^n}x_{e_1}(t) - \dots \\ &\quad - \frac{1}{\tau^{n-k}}x_{e_k}(t) - x_{e_{n+1}}) \\ &= \frac{1}{\tau^{n+1}}y(t) - \frac{1}{\tau^{n+1}}x_{e_1}(t) - \dots \\ &\quad - \frac{1}{\tau^{n-k+1}}x_{e_k}(t) - \frac{1}{\tau}x_{e_{n+1}}. \end{aligned}$$
  Then the equation (13) is true for all $n \in \mathbb{N}$.

References

[1] M.Sato, "Gain-scheduled output-feedback controllers depending solely on scheduling parameters via parameter-dependent lyapunov functions," *Automatica*, vol. 12, pp. 2786–2790, 2011.

[2] D. G. Luenberger, "Observing the state of a linear system," *IEEE Transactions on Military Electronics,*, vol. 8, no. 2, pp. 74 –80, april 1964.

[3] R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory," *Journal of Basic Engineering*, vol. 83, no. 1, pp. 95–108, 1961.

[4] J. Cédric, J. Masse, and M. Fliess, "Étude préliminaire d'une commande sans modèle pour papillon de moteur ..... A model-free control for an engine throttle: a preliminary study," *Journal Européen des Systèmes Automatisés*, vol. 42, no. 2-3, pp. 337–354, 2008.

[5] M. Mboup, "Parameter estimation for signals described by differential equations," *Applicable Analysis*, vol. 88, no. 1, pp. 29–52, 2009.

[6] J. Slotine, "Sliding controller design for non-linear systems," *International Journal of Control*, vol. 40, no. 2, pp. 421–434, 1984.

[7] M. N. Nounou, H. N. Nounou, and M. S. Mahmoud, "Robust adaptive sliding-mode control for continuous time-delay systems," *IMA Journal of Mathematical Control and Information*, vol. 24, no. 3, pp. 299–313, 2007.

[8] X. Yan, S. Spurgeon, and C. Edwards, "Robust sliding mode observer with parameters estimation for a class of nonlinear time-delay systems," in *49th IEEE Conference on Decision and Control (CDC)*, december 2010, pp. 96 –101.

[9] Y. Feng and L. Wang, "Terminal sliding mode observer based parameter estimation method for permanent magnet synchronous motor control system," *International Workshop on Variable Structure Systems*, pp. 184–189, 2010.

[10] E. F. S. Boyd, L. El-Ghaoui and V. Balakrishnan, "Linear matrix inequalities in system and control theory," *SIAM*, vol. 15, pp. 23–24, 1994.