On ISS and iISS properties of homogeneous systems

Emmanuel Bernuau, Andrey Polyakov, Denis Efimov and Wilfrid Perruquetti

***Abstract*—Several conditions are proposed to check input-to-state stability (ISS) and integral input-to-state stability (iISS) properties for generic nonlinear systems applying the weighted homogeneity concept (global or local). The advantages of this result is that, under some mild conditions, the system robustness can be established as a function of the degree of homogeneity.**

I. Introduction

The problem of robustness and stability analysis with respect to external inputs (like exogenous disturbances or measurement noises) for dynamical systems is in the center of attention of many researches [1], [2], [3], [4], [5], [6]. One of the most popular theories, which can be used for this robustness analysis for nonlinear systems, was originated more than 20 years ago [7] and it is based on the Input-to-State Stability (ISS) property and many related notions (see a recent survey [8]). The advantages of ISS theory include a complete list of necessary and sufficient conditions, existence of the Lyapunov method extension, a rich variety of stability concepts adopted for different control and estimation problems.

The main tool to check the ISS property for a nonlinear system consists in a Lyapunov function design, which satisfies some sufficient conditions. As usual, there is no generic approach to select a Lyapunov function for nonlinear systems. Therefore, computationally tractable approaches for ISS verification for particular classes of nonlinear systems are of great practical importance. In this work we are going to propose such a technique for checking ISS and iISS properties for a class of homogeneous and locally homogeneous systems.

Homogeneity is an intrinsic property of an object, which remains consistent with respect to some scaling, e.g. level sets (resp. solutions) are preserved for homogeneous functions (resp. vector fields). The notion of weighted homogeneity was found useful by many authors [9], [10], [11], [12], [13], [14]. The main feature of this property is that it transforms a local (stability) property of the system to the whole state space via a suitably defined scaling. In some cases such a globality of the system behavior becomes ambiguous, that is why the local homogeneity notion has been recently proposed [15], [16]. In this case the property transfer can be carried out on a subspace using different local scales.

The ISS notion of homogeneous systems has been studied in [17], [18], [15]. In this work we are going to generalize the result of those works and extend it to the integral ISS (iISS) property. The underlying idea of the proposed results is that for a nonlinear system its asymptotic stability with zero disturbance implies a certain robustness (ISS or iISS) under homogeneity conditions. Note that to establish asymptotic stability of a homogeneous system one can use a non-strict Lyapunov function with the Krasovskiy–LaSalle arguments.

The outline of the paper is as follows. Notations used in the paper are given in Section II. The robust stability notions under consideration and homogeneity are introduced in Section III. The ISS and iISS properties of homogeneous systems are studied in Section IV. The same analysis for locally homogeneous systems is done in Section V.

II. Notation

Through the paper the following notation is used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real number.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^n$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define the norm $||d||_{[t_0,t_1)} = \text{ess}\ \sup_{t\in[t_0,t_1)} \|d(t)\|$, then $||d||_\infty = ||d||_{[0,+\infty)}$ and the set of $d(t)$ with the property $||d||_\infty < +\infty$ we further denote as $\mathcal{L}_\infty$ (the set of essentially bounded measurable functions).
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t\to+\infty} \beta(s, t) = 0$ for each fixed $s \in \mathbb{R}_+$. For any $\alpha \in \mathcal{K}$ and $s, r \in \mathbb{R}_+$, $\alpha(s + r) \leq \alpha(2s) + \alpha(2r)$.

This work was supported by EU Interreg IV A 2 Mers Seas Zeeën Cross-border Cooperation Programme under SYSIASS project 06-020. It was also supported by Ministry of Higher Education and Research, Nord-Pas de Calais Regional Council and FEDER through the "Contrat de Projets Etat Region (CPER) CIA 2007-2013".

The first and the last two authors are with LAGIS UMR 8219, Université Lille Nord de France, Ecole Centrale de Lille, Avenue Paul Langevin, BP 48, 59651 Villeneuve d'Ascq, France. The last three authors are with Non-A project at INRIA – LNE, Parc Scientifique de la Haute Borne, 40 avenue Halley, Bât. A Park Plaza, 59650 Villeneuve d'Ascq, France. The third author is with the Department of Control Systems and Informatics, Saint Petersburg State University of Information Technologies Mechanics and Optics (ITMO), Kronverkskiy av. 49, Saint Petersburg, 197101, Russia.

- The notation $DV(x)f(x)$ stands for the directional derivative of a continuously differentiable function V with respect to the vector field f evaluated at point x.
- Young's inequality: $sv \leq \frac{s^p}{p} + (1-\frac{1}{p})v^{\frac{p}{p-1}}$ for any $s, v \in \mathbb{R}_+$ and $p > 1$.
- A series of integers $1, 2, ..., n$ is denoted by $\overline{1, n}$.
- $\lfloor x \rceil^\alpha = |x|^\alpha \text{sign}(x)$ for any $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$.

III. Preliminaries

In this work the following nonlinear system is considered:

$$\dot{x} = f(x, d), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^m$ is the external input/disturbance, $d(t) \in \mathcal{L}_\infty$, and $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a locally Lipschitz (or Hölder) continuous function, $f(0, 0) = 0$. In some cases the system (1) is equipped with an output $y \in \mathbb{R}^p$:

$$y = h(x), \tag{2}$$

where $h : \mathbb{R}^n \to \mathbb{R}^p$ is a continuous function. For an initial condition $x_0 \in \mathbb{R}^n$ and input $d \in \mathcal{L}_\infty$, define the corresponding solutions by $x(t, x_0, d)$ for any $t \geq 0$ for which the solution exists.

A. Stability properties

In this work we will be interested in the following stability properties [8].

Definition 1. The system (1) is called *input-to-state practically stable (ISpS)*, if for any input $d \in \mathcal{L}_\infty$ and any $x_0 \in \mathbb{R}^n$ there are some functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $c \geq 0$ such that

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(||d||_{[0,t)}) + c \quad \forall t \geq 0.$$

The function γ is called the *nonlinear asymptotic gain*. The system is called *ISS* if $c = 0$.

Definition 2. The system (1) is called *iISS*, if there are some functions $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty$ the estimate holds:

$$\alpha(\|x(t, x_0, d)\|) \leq \beta(\|x_0\|, t) + \int_0^t \gamma(\|d(s)\|)\, ds \quad \forall t \geq 0.$$

These properties have the following Lyapunov function characterizations.

Definition 3. A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called *ISpS Lyapunov function* for the system (1) if for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and some $r \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\theta \in \mathcal{K}$:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$
$$DV(x)f(x, d) \leq r + \theta(\|d\|) - \alpha_3(\|x\|).$$

Such a function V is called *ISS Lyapunov function* if $r = 0$, and it is *iISS Lyapunov function* if instead $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive definite function.

Note that an ISS Lyapunov function can also satisfy the following equivalent condition for some $\chi \in \mathcal{K}$:

$$\|d\| \leq \chi(\|x\|) \Rightarrow DV(x)f(x, d) \leq -\alpha_3(\|x\|).$$

Theorem 1. [8], [4] *The system (1) is ISS (ISpS, iISS) iff it admits an ISS (ISpS, iISS) Lyapunov function.*

Note that, if the system (1) is ISS, then it is also iISS.

B. Weighted homogeneity

Following [19], for any strictly positive numbers λ and r_i, $i = \overline{1, n}$ called weights, one can define:

- the *vector of weights* $\mathbf{r} = (r_1, \ldots, r_n)^T$, $r_{\max} = \max_{1 \leq j \leq n} r_j$ and $r_{\min} = \min_{1 \leq j \leq n} r_j$;
- the *dilation matrix* $\Lambda_r = diag\{\lambda^{r_i}\}_{i=1}^n$, note that for any $x \in \mathbb{R}^n$ we have $\Lambda_r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_i} x_i, \ldots, \lambda^{r_n} x_n)^T$;
- the *r–homogeneous norm* $\|x\|_r = (\sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}})^{\frac{1}{\rho}}$ for any $x \in \mathbb{R}^n$ and some $\rho > 0$;
- the *unit sphere in the homogeneous norm* $S_r = \{x \in \mathbb{R}^n : \|x\|_r = 1\}$.

Definition 4. A function $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$*–homogeneous with degree* $\mu \in \mathbb{R}$ if for all $x \in \mathbb{R}^n$ we have:

$$\lambda^{-\mu} g(\Lambda_r x) = g(x).$$

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbf{r}$*–homogeneous with degree* $\nu \in \mathbb{R}$, with $\nu \geq -r_{\min}$ if for all $x \in \mathbb{R}^n$ we have:

$$\lambda^{-\nu} \Lambda_r^{-1} f(\Lambda_r x) = f(x),$$

which is equivalent for i-th component of f being a $\mathbf{r}$–homogeneous function of degree $r_i + \nu$.

The system (1) with $d = 0$ is $\mathbf{r}$–homogeneous of degree ν if the vector field f is $\mathbf{r}$–homogeneous of degree ν.

Theorem 2. *[13] For the system (1) with $d = 0$ and $\mathbf{r}$–homogeneous and continuous function f the following properties are equivalent:*

- *the system (1) is (locally) asymptotically stable*;
- *there exists a continuously differentiable $\mathbf{r}$–homogeneous Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$
$$DV(x)f(x, 0) \leq -\alpha(\|x\|),$$
$$\lambda^{-\mu} V(\Lambda_r x) = V(x),\ \mu > r_{\max},$$

for all $x \in \mathbb{R}^n$, for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{K}$.

The $\mathbf{r}$–homogeneity concept presented in Definition 4 is introduced for some $\mathbf{r}$ and all $\lambda > 0$. Restricting the set of admissible values for λ we can introduce local homogeneity [15], [16].

Definition 5. A function $g : \mathbb{R}^n \to \mathbb{R}$ is $(\mathbf{r}_0, \lambda_0, g_0)$–*homogeneous with degree* $\nu_0 \in \mathbb{R}$ (g_0 is a $\mathbf{r}_0$–homogeneous function and $\lambda_0 \in \mathbb{R}_+ \cup \{+\infty\}$) if for all $x \in S_{r_0}$ we have:

$$\lim_{\lambda \to \lambda_0} \left(\lambda^{-\nu_0} g(\Lambda_{r_0} x) - g_0(x) \right) = 0,$$

uniformly on S_{r_0} for $\lambda_0 \in \{0, +\infty\}$.

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is $(\mathbf{r}_0, \lambda_0, f_0)$–*homogeneous with degree* $\mu_0 \geq -r_{0\,\min}$ (f_0 is a $\mathbf{r}_0$–homogeneous vector field and $\lambda_0 \in \mathbb{R}_+ \cup \{+\infty\}$) if for all $x \in S_{r_0}$ we have:

$$\lim_{\lambda \to \lambda_0} \left(\lambda^{-\mu_0} \Lambda_{r_0}^{-1} f(\Lambda_{r_0} x) - f_0(x) \right) = 0,$$

uniformly on S_{r_0} for $\lambda_0 \in \{0, +\infty\}$.

The system (1) for $d = 0$ is $(\mathbf{r}_0, \lambda_0, f_0)$–*homogeneous with degree* $\mu_0 \in \mathbb{R}$ if the vector field f is $(\mathbf{r}_0, \lambda_0, f_0)$–homogeneous with degree μ_0.

The coefficients $r_{0i} > 0$, $i \in \overline{1, n}$ are called weights, ν_0 (respectively μ_0) is the degree of homogeneity (it may depend on λ_0) and g_0 (respectively f_0) is the approximating function of g (respectively f) at λ_0.

Theorem 3. *[20], [13] Let the system (1) with $d = 0$ be ($\mathbf{r}$,0,f_0)–homogeneous with a continuous $f_0 : \mathbb{R}^n \to \mathbb{R}^n$. If the system $\dot{x} = f_0(x)$ is (locally) asymptotically stable, then the system (1) is also locally asymptotically stable.*

Theorem 4. *[15] Let the system (1) with $d = 0$ be ($\mathbf{r}$,$+\infty$,f_∞)–homogeneous with a continuous $f_\infty : \mathbb{R}^n \to \mathbb{R}^n$. If the system $\dot{x} = f_\infty(x)$ is (globally) asymptotically stable, then there exists a compact invariant set $X_\infty \subset \mathbb{R}^n$ containing the origin such that the system (1) is globally asymptotically stable with respect to the set X_∞.*

Lemma 1. *Let a function $g : \mathbb{R}^n \to \mathbb{R}$ be ($\mathbf{r}$,$+\infty$,g_∞)–homogeneous with degree ν, g and g_∞ be locally Lipschitz continuous functions, then for all $x \in \mathbb{R}^n$*

$$|g(x) - g_\infty(x)| \leq \omega(\|x\|_r),\ \omega(s) = \begin{cases} k\, s^{w_{\min}} & \text{if } s \leq 1 \\ k\, s^{w_{\max}} & \text{if } s > 1 \end{cases},$$

where $k > 0$ and $0 \leq w_{\min} \leq \nu$, $0 \leq w_{\max} < \nu$.

Proofs of all results are skipped due to space limitation.

Clearly, it is always possible to select the powers in a way that $0 \leq w_{\min} \leq w_{\max} < \nu$.

Lemma 2. *Let a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ be ($\mathbf{r}$,$+\infty$,f_∞)–homogeneous with degree ν, f and f_∞ be locally Lipschitz continuous, then for all $x \in \mathbb{R}^n$*

$$\|f(x) - f_\infty(x)\| \leq \omega(\|x\|_r),\ \omega(s) = \begin{cases} k\, s^{w_{\min}} & \text{if } s \leq 1 \\ k\, s^{w_{\max}} & \text{if } s > 1 \end{cases},$$

where $k > 0$ and $0 \leq w_{\min} \leq w_{\max} < r_{\max} + \nu$.

IV. Robustness of homogeneous systems

The ISS property of a $\mathbf{r}$–homogeneous system (1) with degree $\nu > 1$ has been investigated in [18], the ISS property of a $\mathbf{r}$–homogeneous system of the form

$$\dot{x} = f_0(x) + G_0(x) d \tag{3}$$

for any admissible degree $\nu \geq -r_{\min}$ (with homogeneous f_0 and G_0) has been studied in [17]. In this work we would like to propose the conditions of ISS and iISS properties for a $\mathbf{r}$–homogeneous system (1) with any $\nu \geq -r_{\min}$.

Define $\tilde{f}(x, d) = [f(x, d)^T \ \ 0_m]^T \in \mathbb{R}^{n+m}$, it is an extended auxiliary vector field for the system (1), where 0_m is the zero vector of dimension m.

Theorem 5. *Let the vector field $\tilde{f}$ be homogeneous with the weights $\mathbf{r} = [r_1, \ldots, r_n] > 0$, $\tilde{\mathbf{r}} = [\tilde{r}_1, \ldots, \tilde{r}_m] \geq 0$ with a degree $\nu \geq -r_{\min}$, i.e. $f(\Lambda_r x, \Lambda_{\tilde{r}} d) = \lambda^\nu \Lambda_r f(x, d)$. Assume that the system (1) is globally asymptotically stable for $d = 0$, then the system (1) is*

ISS *if $\tilde{r}_{\min} > 0$, where $\tilde{r}_{\min} = \min_{1 \leq j \leq m} \tilde{r}_j$;*
iISS *if $\tilde{r}_{\min} = 0$ and $\nu \leq 0$.*

As we can conclude from this result, for the homogeneous system (1) its robustness (ISS or iISS property) is a function of its degree of homogeneity.

Corollary 1. *Let a locally Lipschitz continuous function $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathbf{r}$–homogeneous with a degree ν and asymptotically stable.*

If $f(x, d) = f_0(x) + d$, i.e. d is an additive disturbance, then the system (1) is ISS for $\nu > -r_{\min}$, and iISS for $\nu = -r_{\min}$.

If $f(x, d) = f_0(x + d)$, i.e. d is a measurement noise, then the system (1) is always ISS.

Thus to verify robustness of a system with respect to an external input it is enough to establish its asymptotic stability for the case $d = 0$ and compute its degree of homogeneity performing some algebraic operations, which is a big advantage of the homogeneity approach. In particular the second result of this corollary is related with the problem of state feedback design, which is robust/ISS with respect to measurement noise well studied in [21], [22], [23], [24]. An alternative answer that gives this corollary on the problem stated in [21], [22], [23], [24] is that the closed loop system has to be homogeneous in order to ensure its robustness with respect to measurement noise. An example is the control proposed in [25] for the double integrator (see also [26]):

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u, \quad u = -k_1 \lfloor x_1 \rceil^{\frac{\alpha}{2-\alpha}} - k_2 \lfloor x_2 \rceil^{\alpha},$$

where $k_1 > 0$, $k_2 > 0$ are the control gains to be designed, $\alpha \in (0, 1)$ is a tuning parameter. The closed-loop system is homogeneous for $r_1 = \frac{2-\alpha}{\alpha}$, $r_2 = \frac{1}{\alpha}$ with degree $1 - \frac{1}{\alpha}$. Take a Lyapunov function $V(x_1, x_2) = (1 - \frac{\alpha}{2}) k_1 |x_1|^{\frac{2}{2-\alpha}} + 0.5 x_2^2$, then $\dot{V} = -k_2 |x_2|^{\alpha+1}$ and the system is stable, Krasovskii–LaSalle principle says in this case that the origin is globally attractive, thus the closed-loop system is globally asymptotically stable. Actually, the system is homogeneous with a negative degree, therefore the system is finite-time stable [25], [27]. The result of Corollary 1 claims that the system is also ISS with respect to the measurement noise.

However, the sole homogeneity of $\tilde{f}$ is not enough to claim iISS (ISS), and the case $\tilde{r}_{\min} = 0$ with $\nu > 0$ is the only exclusion as in the following example for $\tilde{\mathbf{r}} = 0$ and $\mathbf{r} = 1$:

$$\dot{x} = (d-1)x^{\alpha},\ \alpha > 1.$$

The asymptotically stable system (1) for $d = 0$ is finite-time stable [28] if it is homogeneous with negative degree [27]. Interestingly to note that the finite-time stability and iISS have a similar restriction on the degree of homogeneity (it has to be negative or non positive for iISS), thus the finite-time stability of a homogeneous system implies iISS.

Corollary 2. *Let the vector field $\tilde{f}$ be homogeneous with the weights $\mathbf{r} = [r_1, \ldots, r_n] > 0$, $\tilde{\mathbf{r}} = [\tilde{r}_1, \ldots, \tilde{r}_m] \geq 0$ with a degree $0 > \nu \geq -r_{\min}$ and asymptotically stable for $d = 0$, then (1) is iISS.*

Theorem 5 also provides a quantitative estimate on the asymptotic gain of (1) in the ISS case.

Corollary 3. *Let the system (1) be ISS under conditions of Theorem 5 and $\nu + \mu > \tilde{r}_{\min}\varrho_{\max}$, then its asymptotic gain γ admits the estimate:*

$$\gamma(s) \leq \ell \begin{cases} s^{\frac{\varrho_{\min}}{\nu+\mu}} & \text{if } s \leq 1 \\ s^{\tilde{r}_{\min}^{-1}} & \text{if } s > 1 \end{cases},\ \ell > 0.$$

The case $\tilde{r}_{\min} = 0$ is critical for Theorem 5, it is possible that the system (1) is ISS while $\tilde{r}_{\min} = 0$ as it is shown in the following example:

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2^2 d_1, \\ \dot{x}_2 &= -x_2^{7/3} + |x_1|^{1/2} d_2, \end{aligned} \tag{4}$$

where $\mathbf{r} = [1\ 1.5]$, $\tilde{\mathbf{r}} = [0\ 3]$, $\nu = 2$ and its ISS Lyapunov function is $V(x) = 0.5x_1^2 + 0.5x_2^2$.

The conditions of Theorem 5 can be technically relaxed skipping homogeneity of $\tilde{f}$ (homogeneity with respect to d). It is worth stressing that homogeneity of $\tilde{f}$ is not a restrictive condition since d is an external input, and we can modify dimension or introduce nonlinear change of coordinates for d.

Theorem 6. *Assume that the system (1) is globally asymptotically stable for $d = 0$ and $\mathbf{r}$–homogeneous with a degree $\nu \geq -r_{\min}$, i.e. $f(\Lambda_r x, 0) = \lambda^{\nu}\Lambda_r f(x, 0)$. Let also for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$*

$$\|f(x,d) - f(x,0)\| \leq \theta(\|x\|_r)\psi(\|d\|) + \phi(\|d\|),$$

$$\theta(s) = \begin{cases} s^{\vartheta_{\min}} & \text{if } s \leq 1 \\ s^{\vartheta_{\max}} & \text{if } s > 1 \end{cases},\ \vartheta_{\min} \geq 0,\ \vartheta_{\max} \in \mathbb{R}$$

for some $\psi, \phi \in \mathcal{K}$. Then the system (1) is

ISS *if $\nu > \vartheta_{\max} - r_{\min}$;*
iISS *if $\nu = \vartheta_{\max} - r_{\min} \leq 0$.*

The result of Theorem 6 can be applied for a larger class of systems, which are not necessarily homogeneous (the function $\tilde{f}$ may be non homogeneous). For example, to the system (3) with non homogeneous G_0 (the result of [17] cannot be used in this case):

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 d_1/(1 + |x_2|), \\ \dot{x}_2 &= -x_2 + x_1^{1/3} d_2, \end{aligned}$$

where $\mathbf{r} = [1\ 1]$ and $\nu = 0$ for $d = 0$, $\vartheta_{\min} = \vartheta_{\max} = 1/3$. However, the conditions obtained in Theorem 6 also do not work for the critical case example (4), where $\vartheta_{\min} = 0.5$, $\vartheta_{\max} = 3$ and the equality $\nu = \vartheta_{\max} - r_{\min}$ is satisfied. A reason of that is hidden in the conservatism of the function θ computation. Another explanation of this fact is that, in the case $\tilde{r}_{\min} = 0$ the system (1) *may not admit* a $\mathbf{r}$–homogeneous ISS Lyapunov function (both theorems 5 and 6 are based on an ISS Lyapunov function of that type), see also the case of Proposition 1 below, where this hypothesis is proven for the case $\tilde{\mathbf{r}} = 0_m$.

Proposition 1. *Considering d as a constant, let the vector field f be $\mathbf{r}$–homogeneous with a degree $\nu \geq -r_{\min}$ independently of d, i.e. $f(\Lambda_r x, d) = \lambda^{\nu}\Lambda_r f(x, d)$ for any $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$. Assume that the system (1) is ISS, then there is* no *$\mathbf{r}$–homogeneous ISS Lyapunov functions for (1).*

Note that an iISS Lyapunov function cannot be homogeneous since α_3 is a bounded positive definite function in this case (if the system is not ISS, see Definition 3), while from consideration above α_3 is proportional to $\|x\|_r^{\nu+\mu}$ for a homogeneous function V. However, the case of Proposition 1 still can be useful for the iISS property.

Theorem 7. *Let the vector field f be $\mathbf{r}$–homogeneous with a degree $0 \geq \nu \geq -r_{\min}$ considering d as a constant, i.e. $f(\Lambda_r x, d) = \lambda^{\nu}\Lambda_r f(x, d)$ for any $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$. Then the system (1) is globally asymptotically stable for $d = 0$ iff it is iISS.*

This result can be applied, for example, to "bilinear" systems:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x d_i), \tag{5}$$

where all f_i, $i = \overline{0, m}$ are $\mathbf{r}$–homogeneous functions of the same degree with respect to x, $f_i(0) = 0$ (a simplest example is $f_i(x) = A_i x$, where $A_i \in \mathbb{R}^{n \times n}$). According to Theorem 7, if in (5) the system $\dot{x} = f_0(x)$ is asymptotically stable and the homogeneity degree is non-positive, then the system is iISS.

To finish comparison of theorems 5 and 6 note that the conditions of Theorem 6 may be more restrictive than in

Theorem 5, as it can be demonstrated in the following example:

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2^{1/3} d_1,\\ \dot{x}_2 &= -x_2^{5/3} + x_1^3 d_2,\end{aligned}$$

where $\mathbf{r} = [1\ 3]$, $\tilde{\mathbf{r}} = [2\ 2]$, $\nu = 2$ and it is ISS by Theorem 5 (it also has a homogeneous ISS Lyapunov function $V(x) = x_1^6/6 + x_2^2/2$), but Theorem 6 does not work since $\vartheta_{\min} = 1$, $\vartheta_{\max} = 3$ and $\nu = \vartheta_{\max} - r_{\min}$. In addition, the iISS condition in Theorem 6 implicitly needs $\vartheta_{\max} < r_{\min}$. Another interpretation of the ISS condition of Theorem 6 is that the system (1) has local approximation at infinity $f(x, 0)$.

V. Robustness of locally homogeneous systems

The ISS property of locally homogeneous systems has been analyzed in [15], it was shown there that if the system (1) is locally homogeneous at 0 and $+\infty$, and all approximations and the system itself are globally asymptotically stable for $d = 0$, then (1) is ISS. First we are going to propose a variant of that proof for approximation at infinity and, next, we will extend it for the systems not homogeneous with respect to d.

Now assume that the system is locally homogeneous at infinity. Define $\tilde{f}_\infty(x, d) = [f_\infty^T(x, d)\ 0_m^T]^T$.

Assumption 1. *Let the vector field $\tilde{f}$ be $((\mathbf{r}, \tilde{\mathbf{r}}), +\infty, \tilde{f}_\infty)$–homogeneous with the weights $\mathbf{r} = [r_1, \dots, r_n] > 0$, $\tilde{\mathbf{r}} = [\tilde{r}_1, \dots, \tilde{r}_m] > 0$ and degree $\nu > -r_{\min}$, i.e. for any $\epsilon > 0$ there is a $\lambda_\epsilon > 0$ such that $\sup_{\lambda \geq \lambda_\epsilon} \|\lambda^{-\nu} \Lambda_r^{-1} f(\Lambda_r y, \Lambda_{\tilde{r}} d) - f_\infty(y, d)\| \leq \epsilon$ for all $y \in S_r$ and $d \in S_{\tilde{r}}$, where f_∞ is a locally Lipschitz continuous function.*

Since $\|\tilde{f}(x, d) - \tilde{f}_\infty(x, d)\| = \|f(x, d) - f_\infty(x, d)\|$, define $g(x, d) = f(x, d) - f_\infty(x, d)$, then by Lemma 1 in this case for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$ we have

$$\begin{aligned}|g_i(x, d)| &\leq \omega_i(\|x\|_r + \|d\|_{\tilde{r}}),\\ \omega_i(s) &= \begin{cases} k\, s^{w_{\min}^i} & \text{if } s \leq 1 \\ k\, s^{w_{\max}^i} & \text{if } s > 1 \end{cases}\end{aligned}$$

for all $i = \overline{1, n}$, where $k > 0$, $w_{\min}^i \geq 0$ and $w_{\max}^i = r_i + \nu - \delta < r_i + \nu$ for some $\delta > 0$.

Theorem 8. *Let Assumption 1 be satisfied. Assume that the system $\dot{x} = f_\infty(x, d)$ is globally asymptotically stable for $d = 0$, then the system (1) is ISpS.*

For an example, consider the system:

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1^3 + x_2 |x_1|^{0.75} d,\\ \dot{x}_2 &= x_2 - |x_2| x_2 + |x_1|^{3.5} |x_2|^{0.125} d,\end{aligned}$$

which is $((\mathbf{r}, \tilde{\mathbf{r}}), +\infty, \tilde{f}_\infty)$–homogeneous with the weights $\mathbf{r} = [1, 2]$, $\tilde{\mathbf{r}} = 0.25$ and degree $\nu = 2$ with $f_\infty(x, d) = [-x_1^3 + x_2|x_1|^{0.75} d \quad -|x_2|x_2 + |x_1|^{3.5}|x_2|^{0.125} d]^T$. The linearization of the system is unstable and it is hard to simulate this system in order to check its stability since it is very stiff. However, since all conditions of Theorem 8 are satisfied, the system is ISpS.

Corollary 4. *Let a locally Lipschitz continuous vector field $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ be $(\mathbf{r}, +\infty, f_\infty)$–homogeneous with a degree $\nu > -r_{\min}$ and asymptotically stable. If $f(x, d) = f_0(x) + d$, i.e. d is an additive disturbance, or $f(x, d) = f_0(x + d)$, i.e. d is a measurement noise, then the system (1) is ISpS.*

This corollary completes Corollary 1, indicating a solution to the problem of feedback design that is ISS/ISpS with respect to measurement noise posed in [21], [22], [23], [24]. There is a modification of Theorem 8, which skips homogeneity with respect to d in Assumption 1.

Assumption 2. *Let the vector field f be $(\mathbf{r}, +\infty, f_\infty)$–homogeneous with degree $\nu > -r_{\min}$ for $d = 0$, i.e. for any $\epsilon > 0$ there is a $\lambda_\epsilon > 0$ such that $\sup_{\lambda \geq \lambda_\epsilon} \|\lambda^{-\nu} \Lambda_r^{-1} f(\Lambda_r y, 0) - f_\infty(y, 0)\| \leq \epsilon$ for all $y \in S_r$, where f_∞ is a locally Lipschitz continuous function. Let also for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$ the inequality*

$$\|f(x, d) - f(x, 0)\| \leq \theta(\|x\|_r)\psi(\|d\|) + \phi(\|d\|),$$

$$\theta(s) = \begin{cases} s^{\vartheta_{\min}} & \text{if } s \leq 1 \\ s^{\vartheta_{\max}} & \text{if } s > 1 \end{cases},\ \vartheta_{\min} \geq 0,\ \vartheta_{\max} \in \mathbb{R}$$

be satisfied for some $\psi, \phi \in \mathcal{K}$ and $\nu > \vartheta_{\max} - r_{\min}$.

Define $g(x) = f(x, 0) - f_\infty(x, 0)$, by Lemma 1 in this case for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned}|g_i(x)| &\leq \omega_i(\|x\|_r),\\ \omega_i(s) &= \begin{cases} k\, s^{w_{\min}^i} & \text{if } s \leq 1 \\ k\, s^{w_{\max}^i} & \text{if } s > 1 \end{cases},\end{aligned}$$

where $k > 0$, $w_{\min}^i \geq 0$ and $w_{\max}^i = r_i + \nu - \delta < r_i + \nu$ for some $\delta > 0$.

Theorem 9. *Let Assumption 2 be satisfied. Assume that the system $\dot{x} = f_\infty(x, 0)$ is globally asymptotically stable, then the system (1) is ISpS.*

Theorems 8 and 9 extend the conditions of theorems 5 and 6 on the case of local homogeneity at infinity. However, in the local case the difference between applicability conditions of theorems 8 and 9 is minor, the main advantage is that the local approximation at infinity may be failed to exist for both variables x and d (the case of Theorem 8), but it may exist for $d = 0$ and Theorem 9 can be applied int this case.

VI. Conclusion

Several conditions of the ISS and iISS properties have been developed based on the homogeneity theory. The advantage of these conditions is that the system robustness can be checked after its asymptotic stability in the unperturbed case provided that some algebraic homogeneity constraints are satisfied for the system equations (globally or locally). All results are obtained for generic nonlinear systems. Several examples are proposed showing efficiency of the proposed theory and its limitations.

REFERENCES

[1] J. Doyle, B. Francis, and A. Tannenbaum, *Feedback Control Systems*. Mac Millan Publishing Co, 1992.

[2] D. J. Hill and P. J. Moylan, "Dissipative dynamical systems: basic input-output and state properties," *J. Franklin Inst.*, vol. 309, no. 5, pp. 327–357, 1980.

[3] A. van der Schaft, *L_2-gain and passivity techniques in nonlinear control*, vol. 218 of *Lecture Notes in Control and Information Sciences*. London: Springer-Verlag London Ltd., 1996.

[4] E. D. Sontag, "The ISS philosophy as a unifying framework for stability-like behavior," in *Nonlinear control in the year 2000, Vol. 2 (Paris)*, vol. 259 of *Lecture Notes in Control and Inform. Sci.*, pp. 443–467, London: Springer, 2001.

[5] M. Vidyasagar, *Input-output analysis of large-scale interconnected systems*, vol. 29 of *Lecture Notes in Control and Information Sciences*. Berlin: Springer-Verlag, 1981. Decomposition, well-posedness and stability.

[6] J. C. Willems, "Dissipative dynamical systems. I. General theory," *Arch. Rational Mech. Anal.*, vol. 45, pp. 321–351, 1972.

[7] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Control*, vol. 34, no. 4, pp. 435–443, 1989.

[8] S. Dashkovskiy, D. Efimov, and E. Sontag, "Input to state stability and allied system properties," *Automation and Remote Control*, vol. 72, no. 8, pp. 1579–1614, 2011.

[9] S. Bhat and D. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Mathematics of Control, Signals and Systems*, vol. 17, pp. 101–127, 2005.

[10] L. Grüne, "Homogeneous state feedback stabilization of homogeneous systems," *SIAM J. Control Optimization*, vol. 38, no. 4, pp. 1288–1314, 2000.

[11] H. Hermes, "Homogeneous feedback controls for homogeneous systems," *Systems & Control letters*, vol. 24, pp. 7–11, 1995.

[12] M. Kawski, "Nilpotent Lie algebras of vector fields," *J. reine angew. Math*, p. 1, 1988.

[13] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector field," *Systems & Control Letters*, vol. 19, pp. 467–473, 1992.

[14] R. Sepulchre and D. Aeyels, "Stabilizability does not imply homogeneous stabilizability for controllable systems," *SIAM J. Control Optimization*, vol. 34, no. 5, pp. 1798–1813, 1996.

[15] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous Approximation, Recursive Observer Design, and Output Feedback," *SIAM J. Control Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.

[16] D. Efimov and W. Perruquetti, "Oscillations conditions in homogenous systems," in *Proc. IFAC NOLCOS Symp.*, pp. 1379–1384, 2010.

[17] Y. Hong, "H_∞ control, stabilization, and input-output stability of nonlinear systems with homogeneous properties," *Automatica*, vol. 37, no. 7, pp. 819–829, 2001.

[18] E. Ryan, "Universal stabilization of a class of nonlinear systems with homogeneous vector fields," *Systems & Control Letters*, vol. 26, pp. 177–184, 1995.

[19] A. Bacciotti and L. Rosier, *Lyapunov Functions and Stability in Control Theory*. Springer, 2nd ed., 2005.

[20] H. Hermes, "Nilpotent approximations of control systems and distributions," *SIAM J. Control Optim.*, vol. 24, p. 731, 1986.

[21] R. Freeman, "Global internal stabilizability does not imply global external stabilizability for small sensor disturbances," *IEEE Trans. Automat. Control*, vol. 40, pp. 2119–2122, 1995.

[22] Z.-P. Jiang, I. Mareels, and D. Hill, "Robust control of uncertain nonlinear systems via measurement feedback," *IEEE Trans. Automat. Control*, vol. 44, pp. 807–812, 1999.

[23] R. Sanfelice and A. Teel, "On hybrid controllers that induce input-to-state stability with respect to measurement noise," in *Proc. 44th IEEE Conf. on Decision and Control*, pp. 4891–4896, 2005.

[24] C. Ebenbauer, T. Raff, and F. Allgöwer, "Certainty-equivalence feedback design with polynomial-type feedbacks which guarantee iss," *IEEE Trans. Automat. Control*, vol. 52, pp. 716–720, 2007.

[25] S. P. Bhat and D. Bernstein, "Continuous finite-time stabilization of the translational and rotational double integrator," *IEEE Trans. Automat. Control*, vol. 43, pp. 678–682, may 1998.

[26] E. Bernuau, W. Perruquetti, D. Efimov, and E. Moulay, "Finite-time output stabilization of the double integrator," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pp. 5906–5911, 2012.

[27] S. Bhat and D. Bernstein, "Finite time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[28] W. Perruquetti and S. Drakunov, "Finite time stability and stabilisation," in *IEEE Conference on Decision and Control*, no. 2, (Sydney, Australia), pp. 1894–1899, 2000.