

# On an extension of homogeneity notion for differential inclusions

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**Abstract**—The notion of geometric homogeneity is extended for differential inclusions. This kind of homogeneity provides the most advanced coordinate-free framework for analysis and synthesis of nonlinear discontinuous systems. Theorem of L. Rosier [1] on a homogeneous Lyapunov function existence for homogeneous differential inclusions is presented. An extension of the result of Bhat and Bernstein [2] about the global asymptotic stability of a system admitting a strictly positively invariant compact set is also proved.

## I. INTRODUCTION AND RELATED WORKS

On one hand, the homogeneity is a well established tool for treatment of nonlinear Ordinary Differential Equations (ODE). It is an intrinsic property of an object, which remains consistent with respect to some scaling: level sets (resp. solutions) are preserved for homogeneous functions (resp. vector fields). The homogeneity notion was introduced in Control Theory in order to investigate stability properties (see [3], [4], [5]). The weighted homogeneity was introduced by V.I. Zubov [3], and by H. Hermes [6],[7] when looking at a local approximation of nonlinear system. Asymptotic controllability is shown to be inherited by the original nonlinear system if this property holds for the homogeneous approximation [7], [8], [9]. With this homogeneity property, many results have been obtained for stability/stabilization [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], or output feedback [20]. Other useful extensions are the homogeneity with a degree, which is a function of the state [21], and bi-limit homogeneity [20], which makes homogeneous approximation valid both at the origin and at infinity. Those tools have been useful for nonlinear observer and output feedback designs. Extensions to local homogeneity have been proposed recently [20], [22]. Let us note that this notion was also used in different contexts: polynomial systems [23] and switched systems [24], self-triggered systems [25], control and analysis of oscillations [22], [26].

On the other hand, in many practical situations, one encounters ODE with discontinuous right hand side: for

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example when dealing with variable structure systems, systems with adaptive control, power electronic systems with switching devices, mechanical systems with friction, etc. It is well known that the regularization procedure (due to Filippov) leads to a Differential Inclusion (DI). For example, when investigating sliding mode, in particular higher order sliding mode, a required property is Finite-Time Stability (FTS), which is easily obtained for a locally asymptotically stable homogeneous system when its degree of homogeneity is negative [27]. However the analysis of the homogeneity notion for DIs is not complete. Only few results exist in this context [28], [29], [24]. Unfortunately, the formulations of these definitions include some properties of solutions of discontinuous systems in addition to properties of the set-valued map involved in the right-hand side of the considered DI. Moreover, since homogeneity is a kind of symmetry, it should be invariant under a change of coordinates. This motivates for a geometric, coordinate free definition of homogeneity.

The very first geometric definitions, in the context of ODE, appear independently by V.V. Khomenuk in [30], M. Kawski in [16], [31], [32] and L. Rosier in [33]. Then [2] gives a counterpart to [1] in this context, and [25] uses it for self-triggered systems.

This paper aims at extending the homogeneity notion for differential inclusions and the theorem of L. Rosier in [1] (if a homogeneous system is globally asymptotically stable, then there exists a homogeneous proper Lyapunov function - note that this theorem has also been obtained in [3]). Thus after some notations and preliminaries concerning differential inclusions and stability concepts (Section II), Section III will provide a coordinate-free definition of homogeneity for DIs: it will be shown that such definition is consistent with the well known Filippov regularization procedure. Then, the main results are given in Section IV:

- extension of Rosier's theorem [1] to DIs;
- extensions of the results from [2] concerning FTS and another one relating existence of strictly positively invariant compact set and asymptotic stability.

Finally a conclusion will sum up the paper results and possible extensions.

## II. NOTATIONS AND PRELIMINARIES

### A. Notations

Through the paper  $n$  is a strictly positive integer, and  $|\cdot|$  denotes the euclidian norm on  $\mathbb{R}^n$  as well as the absolute value of a real number, depending on the context. We will denote by  $B(x, \varepsilon) = \{y \in \mathbb{R}^n : |x - y| < \varepsilon\}$  the open ball

and  $\bar{B}(x, \varepsilon)$  the closed ball. The vector space  $\mathbb{R}^n$  is endowed with the Borel  $\sigma$ -algebra and the Lebesgue measure  $\lambda$ . We will denote by  $\mathcal{N}$  the set of all zero measure subsets of  $\mathbb{R}^n$ . For a positive integer  $k$ ,  $\mathcal{C}^k$  is the set of functions having continuous derivatives of the order  $k$  or higher. The notation  $d_x V$  stands for the differential of a differentiable function  $V$  at a point  $x$ . Applying this differential to a vector field  $f$ , we get  $\mathcal{L}_f V(x) = d_x V f(x)$ , i.e. the Lie derivative of  $V$  along  $f$  evaluated at point  $x$ .

If  $A$  is a subset of  $\mathbb{R}^n$ , we denote by  $\overset{\circ}{A}$  the interior of  $A$ , that is the biggest open subset of  $A$ . Similarly, we denote by  $\bar{A}$  the closure of  $A$ , that is the smallest closed set containing  $A$ . The boundary of  $A$  is defined by  $\partial A = \bar{A} \setminus \overset{\circ}{A}$ . Let us recall that  $A$  is convex iff for any  $x, y \in A$ , for all  $t \in [0, 1]$ ,  $tx + (1-t)y \in A$ . We denote by  $\text{conv } A$  the smallest convex set containing  $A$ , and  $\overline{\text{conv}} A$  the smallest closed convex set containing  $A$ .

If  $A$  and  $B$  are bounded subsets of  $\mathbb{R}^n$ , the distance between  $A$  and  $B$  is defined by  $d(A, B) = \inf\{|a - b|, a \in A, b \in B\}$ .

### B. Differential inclusion and stability concepts

Since the aim of the paper is to extend some homogeneity definitions and results from ODE to DI, let us stress some notions and some differences between the two cases.

Solutions: For ODE, there exist classical sufficient conditions ensuring existence (Carathéodory or Peano conditions) and sometimes also uniqueness (Lipschitz or dissipative conditions) of solution(s) for a Cauchy Problem. These conditions depend on the smoothness of function in the right-hand side of the ODE. Thus for ODE, they may possess unique maximal solutions in forward time. However, DI are mainly introduced to capture behaviors with no uniqueness. In that case some *standard assumptions* ensure existence of solutions, which are absolutely continuous functions of the time (see Section III-B). Consider an ODE with discontinuous right hand side of the form ( $\dot{x} \triangleq \frac{dx}{dt}$ ):

$$\dot{x} = f(x) \quad (1)$$

where  $f$  is measurable and locally essentially bounded. To handle this situation, it is worthy to replace it with the following DI:

$$\dot{x} \in F(x), \quad (2)$$

with

$$F(x) = \bigcap_{\varepsilon > 0} \bigcap_{N \in \mathcal{N}} \overline{\text{conv}}(f(y), y \in B(x, \varepsilon) \setminus N). \quad (3)$$

This is the Filippov regularization procedure.

Stability: Among the stability notions that we will deal with, we say that a vector field is LAT (locally attractive), LAS (locally asymptotically stable), GAS (globally asymptotically stable) if the origin is so (we are only considering the origin as a desired equilibrium). All these notions are well known in the case of forward uniqueness of solutions (see [34]). However, let us stress that without this uniqueness assumption, it may happen that starting from an initial

condition some solutions may converge to the origin while others do not. This is the reason why one need to introduce strong or weak properties: strong (respectively weak) means that the property holds for all solutions (respectively at least for one solution) originated from an initial condition. When this distinction is not explicitly mentioned then it means that we are dealing with the strong property.

*Definition 2.1:* We say that a compact set  $K$  is SPI (strictly positively invariant) for a given vector field  $f$  (resp. a set valued map  $F$ ) if every solution of  $\dot{x} = f(x)$  (resp.  $\dot{x} \in F(x)$ ) starting in the boundary of  $K$  at  $t = 0$  belongs to the interior of  $K$  for all  $t > 0$ .

## III. HOMOGENEITY: FROM ORDINARY DIFFERENTIAL EQUATIONS TO DIFFERENTIAL INCLUSIONS

### A. A quick overview of homogeneity for ODE

For the weighted notion of homogeneity, it is possible that a system is not homogeneous using a set of coordinates while it is homogeneous using other coordinates. Thus a coordinate-free definition is required [16], [33]:

*Definition 3.1:* A complete<sup>1</sup> vector field  $\nu : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called Euler if  $\nu$  is  $\mathcal{C}^1$  and  $-\nu$  is GAS. Denote the flow of  $\nu$  by  $\Phi$ , that is  $\Phi^s(x)$  is the current state at time  $s$  of the trajectory of  $\nu$  starting from  $x$  at  $s = 0$ .

*Definition 3.2:* Let  $\nu$  be an Euler vector field. A function  $V$  is said to be  $\nu$ -homogeneous of degree  $k$  iff for all  $s \in \mathbb{R}$  we have:

$$V(\Phi^s(x)) = e^{ks} V(x). \quad (4)$$

A vector field  $f$  is said to be  $\nu$ -homogeneous of degree  $m$  iff for all  $s \in \mathbb{R}$  we have:

$$f(\Phi^s(x)) = e^{ms} d_x \Phi^s f(x). \quad (5)$$

*Example 3.3:* On  $\mathbb{R}^2$ , set  $\nu(x) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x$ . This vector field is Euler, as it can be seen computing its flow  $\Phi^s(x) = e^s R(s)x$ , where  $R(s) = \begin{pmatrix} \cos s & \sin s \\ \sin s & -\cos s \end{pmatrix}$ . Consider now the vector field  $f$  defined by:

$$f(x) = \begin{pmatrix} \cos(\ln|x|) \\ \sin(\ln|x|) \end{pmatrix}.$$

This vector field is continuous on  $\mathbb{R}^2 \setminus \{0\}$ , globally bounded and  $\nu$ -homogeneous of degree  $-1$ . Indeed,  $f(\Phi^s(x)) = R(s)f(x) = e^{-s} d_x \Phi^s f(x)$ .

*Remark 3.4:* In the sequel we will often omit the vector field  $\nu$ , and say “ $f$  is homogeneous” instead of saying “there exists a Euler vector field  $\nu$  such that  $f$  is  $\nu$ -homogeneous”.

An essential feature of a homogeneous vector field is usually referred as the “flow commutation property”.

*Proposition 3.5:* [16], [33] Let  $f$  be a continuous vector field with forward unicity of solutions and denote its semi-flow by  $\Psi$ . The vector field  $f$  is  $\nu$ -homogeneous of degree  $m$  iff

$$\Phi^s \circ \Psi^{e^{ms}t} = \Psi^t \circ \Phi^s \quad \forall s \in \mathbb{R} \quad \forall t \geq 0. \quad (6)$$

<sup>1</sup>A vector field is complete if its integral curves are defined for all  $t \in \mathbb{R}$ .

The definition of the homogeneity needs to compute the flow  $\Phi$  of  $\nu$ , which is a difficult task in general. That is why there exists an equivalent condition for homogeneity assuming additional regularity properties.

*Proposition 3.6:* [16] Let  $V$  be a  $\mathcal{C}^1$  function and  $f$  be a  $\mathcal{C}^1$  vector field. Then  $V$  (resp.  $f$ ) is  $\nu$ -homogeneous of degree  $m$  iff  $\mathcal{L}_\nu V = mV$  (resp.  $[\nu, f] = mf$ ).

Algebraic operations on homogeneous objects produce homogeneous objects. Indeed, a sum of two homogeneous tensors of the same degree, the differentiation, the Lie derivative, the interior product between homogeneous tensors lead to a homogeneous tensor. The multiplication of a homogeneous tensor of degree  $m$  by a homogeneous function of degree  $k$  leads to a homogeneous tensor of degree  $m+k$ .

Similarly to linear vector fields, homogeneous vector fields have many properties related to stability. Let us recall some of them, which will be used in the sequel. Let  $f$  be a continuous homogeneous vector field of degree  $m$  with the property of forward unicity of solutions (the latter one is not necessary for the third result below). The following implications hold:

- 1) If  $f$  is LAT, then  $f$  is GAS [2].
- 2) If there exists a SPI compact, then  $f$  is GAS [2].
- 3) If  $f$  is continuous and GAS, then there exists a homogeneous proper Lyapunov function for  $f$  of degree  $k$  for all  $k > \max(0, -m)$  [1].

All these stability results hold for continuous dynamics. In the case of a discontinuous vector field, or a differential inclusion, the extensions of homogeneity were provided by:

- A.F. Filippov in [28], but only for standard homogeneity,
- Y. Orlov in [24], but his definition was based on the properties of the trajectories of the system, which are usually unknown,
- A. Levant in [29], but his definition was extended only to the weighted homogeneity notion, which depends on the choice of coordinates.

### B. Homogeneity for differential inclusions

In this section we consider a vector field or the associated differential inclusion given by the Filippov procedure. We will define a homogeneity notion consistent with the current definition and with the nice properties introduced above.

We consider the autonomous differential inclusion defined by the set-valued map  $F$ :

$$\dot{x} \in F(x). \quad (7)$$

An absolute continuous curve  $x$  defined on an interval  $I$  is called a trajectory of (7) if for almost every  $t \in I$ , we have  $\dot{x}(t) \in F(x(t))$ . We say that a trajectory starts at  $x_0$  if  $x$  is defined on an interval containing 0 and  $x(0) = x_0$ . We will denote by  $\mathcal{S}([0, T], A)$  the set of trajectories of (7) defined on the interval  $[0, T]$ ,  $T > 0$ , starting in  $A \subset \mathbb{R}^n$ . We also allow  $T = +\infty$ , and in this situation the interval  $[0, T]$  has to be understood as  $[0, +\infty[$ . We will also denote  $\mathcal{S}([0, T], x_0) = \mathcal{S}([0, T], \{x_0\})$ . Note that without other assumptions, it may

happen that for any  $T \in ]0, +\infty]$  some trajectories of (7) are not defined on  $[0, T]$  (finite-time blow up for any positive time, e.g.  $\dot{x} \in F(x) = \mathbb{R}_+$ ).

Let  $T \in ]0, +\infty]$  be such that every trajectory of (7) starting in  $A$  is defined on  $[0, T]$ . We denote  $\Psi^T(A) = \{x(T) : x \in \mathcal{S}([0, T], A)\}$ . This set is the reachable set from  $A$  at time  $T$ , or the limit in case  $T = +\infty$ . Let us stress that under the unicity in forward time,  $\Psi^t$  corresponds to the semiflow of  $F$ ; this remark justifies that we call  $\Psi$  the generalized flow of  $F$ .

*Definition 3.7:* Let  $\nu$  be a Euler vector field. A set-valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is  $\nu$ -homogeneous of degree  $m \in \mathbb{R}$  if for all  $x \in \mathbb{R}^n$  and for all  $s \in \mathbb{R}$  we have:

$$F(\Phi^s(x)) = e^{ms} d_x \Phi^s F(x).$$

The system (7) is  $\nu$ -homogeneous of degree  $m$  if the set-valued map  $F$  is homogeneous of degree  $m$ .

*Proposition 3.8:* Let  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a set-valued  $\nu$ -homogeneous map of degree  $m$ . Then for all  $x_0 \in \mathbb{R}^n$  and any trajectory  $x$  of the system (7) starting at  $x_0$  and all  $s \in \mathbb{R}$ , the absolute continuous curve  $t \mapsto \Phi^s(x(e^{ms}t))$  is a trajectory of the system (7) starting at  $\Phi^s(x_0)$ .

*Proof:* Consider a trajectory  $x$  of (7) starting at  $x_0$ . The curve  $t \mapsto \Phi^s(x(e^{ms}t))$  is clearly an absolute continuous curve for all  $s \in \mathbb{R}$ . Moreover, for almost all  $t \in \mathbb{R}$  we have:

$$\begin{aligned} \frac{d}{dt} \Phi^s(x(e^{ms}t)) &= e^{ms} d_{x(e^{ms}t)} \Phi^s \dot{x}(e^{ms}t) \\ &\in e^{ms} d_{x(e^{ms}t)} \Phi^s F(x(e^{ms}t)). \end{aligned}$$

Since  $F$  is  $\nu$ -homogeneous of degree  $m$ , we find  $\frac{d}{dt} \Phi^s(x(e^{ms}t)) \in F(\Phi^s(x(e^{ms}t)))$  and thus  $t \mapsto \Phi^s(x(e^{ms}t))$  is a solution of the system (7) for all  $s \in \mathbb{R}$ . ■

*Remark 3.9:* This proposition is the extension of Proposition 3.5. The proposition may also be recast using the generalized flow as:

$$\Psi^t(\Phi^s(A)) = \Phi^s(\Psi^{e^{ms}t}(A)). \quad (8)$$

Now, similarly to the usual setting, a lot of properties can be extended from a sphere to everywhere outside the origin by homogeneity.

*Proposition 3.10:* Let  $F$  be a  $\nu$ -homogeneous set-valued map of degree  $m$ . Then  $F(x)$  is compact for all  $x \in \mathbb{R}^n \setminus \{0\}$  iff  $F(x)$  is compact for all  $x \in \mathbb{S}_r$ , where  $\mathbb{S}_r = \{x \in \mathbb{R}^n : |x| = r\}$ ,  $r > 0$ . The same property holds for convexity or upper semi-continuity<sup>2</sup>.

*Proof:* The result about compactness or convexity is straightforward. Let us only prove that if  $F(x)$  is upper semi-continuous on the sphere, so is  $F$  everywhere outside of the origin.

Set  $y \neq 0$ . There exists  $s \in \mathbb{R}$  and  $x \in \mathbb{S}$  such that  $\Phi^s(x) = y$ . Fix  $\mathcal{V}$  a neighborhood of  $F(y) = F(\Phi^s(x)) = e^{ms} d_x \Phi^s F(x)$ . Eventually replacing  $\mathcal{V}$  by a bounded neighborhood of  $F(y)$  included in  $\mathcal{V}$ , we assume that  $\mathcal{V}$  is

<sup>2</sup>Let  $E_1$  and  $E_2$  be two topological Hausdorff spaces, a set-valued map  $F : E_1 \rightrightarrows E_2$  is **upper semi-continuous** at  $x \in \text{dom}(F)$  if, for any neighborhood  $\mathcal{V} \subset E_2$  of  $F(x)$ , there exists a neighborhood  $\mathcal{U}$  of  $x$  such that  $F(\mathcal{U}) \subset \mathcal{V}$ .

bounded. Consider a bounded neighborhood  $\mathcal{V}_0 \subset \mathcal{V}$  of  $F(y)$  such that there exists  $\alpha > 0$  with  $d(\mathcal{V}_0, \partial\mathcal{V}) \geq \alpha$ , and denote by  $\tilde{\mathcal{V}}_0 = e^{-ms}(d_x\Phi^s)^{-1}\mathcal{V}_0$ .  $\tilde{\mathcal{V}}_0$  is a neighborhood of  $F(x)$ . Let us denote  $M = \sup_{v \in \mathcal{V}_0} |v| > 0$ . Let us also denote by  $\sigma_{max}(d_z\Phi^s(d_x\Phi^s)^{-1})$  the biggest singular value of the linear mapping  $d_z\Phi^s(d_x\Phi^s)^{-1}$ . The function  $\varphi : z \mapsto |\sigma_{max}(d_z\Phi^s(d_x\Phi^s)^{-1}) - 1|$  is continuous and vanishes at  $z = x$ . Therefore, there exists a neighborhood  $\tilde{\mathcal{U}}$  of  $x$  on which  $\varphi(z) < \frac{\alpha}{M}$ . By upper semi-continuity of  $F$  at  $x$ , there exists  $\tilde{\mathcal{U}}_0$  a neighborhood of  $x$  such that for all  $\tilde{z} \in \tilde{\mathcal{U}}_0$ ,  $F(\tilde{z}) \subset \tilde{\mathcal{V}}_0$ . Set  $\mathcal{U} = \Phi^s(\tilde{\mathcal{U}} \cap \tilde{\mathcal{U}}_0)$ , then  $\mathcal{U}$  is a neighborhood of  $y$ . Let  $z$  be an element of  $\mathcal{U}$ . Then there exists  $\tilde{z} \in \tilde{\mathcal{U}} \cap \tilde{\mathcal{U}}_0$  such that  $z = \Phi^s(\tilde{z})$ . Therefore  $F(z) = F(\Phi^s(\tilde{z})) = e^{ms}d_{\tilde{z}}\Phi^s F(\tilde{z}) \subset e^{ms}d_{\tilde{z}}\Phi^s\tilde{\mathcal{V}}_0$  since  $\tilde{z} \in \tilde{\mathcal{U}}_0$ . But  $\tilde{\mathcal{V}}_0 = e^{-ms}(d_x\Phi^s)^{-1}\mathcal{V}_0$ , thus  $F(z) \subset d_z\Phi^s(d_x\Phi^s)^{-1}\mathcal{V}_0$ . Let  $v \in \mathcal{V}_0$  be fixed. We have:

$$|d_z\Phi^s(d_x\Phi^s)^{-1}v - v| \leq \varphi(\tilde{z})M.$$

Since  $\tilde{z} \in \tilde{\mathcal{U}}$ , we find  $|d_z\Phi^s(d_x\Phi^s)^{-1}v - v| \leq \alpha$ , and hence  $d_z\Phi^s(d_x\Phi^s)^{-1}v \in \mathcal{V}$ . Finally we conclude that  $F(z) \subset \mathcal{V}$  and the proposition is proved. ■

In the sequel we will say that  $F$  satisfies the *standard assumptions* if  $F$  is upper semi-continuous, and for all  $x \in \mathbb{R}^n$ ,  $F(x)$  is not empty, compact and convex. It is well known that any locally essentially bounded function gives a multivalued function which satisfies the standard assumptions through the Filippov regularization procedure (see [35] for more details). We will denote by  $\mathcal{L}_{loc}^\infty$  the set of locally essentially bounded vector fields on  $\mathbb{R}^n$ .

Indeed, in many situations, the set-valued map  $F$  comes from the Filippov regularization procedure of a discontinuous vector field  $f$ . Suppose that we have a vector field  $f$ , which is homogeneous in the sense of Definition 3.2. If we apply the regularization procedure, is the homogeneity property preserved? The answer is positive.

*Proposition 3.11:* Let  $f \in \mathcal{L}_{loc}^\infty$  be a vector field and  $F$  be the associated set-valued map. Suppose  $f$  is  $\nu$ -homogeneous of degree  $m$ . Then  $F$  is  $\nu$ -homogeneous of degree  $m$ .

*Example 3.12:* Consider the  $n$ -integrator with an input  $u(x) = -\sum_i k_i \text{sgn}(x_i)$ ,  $k_i > 0$ :

$$\begin{cases} \dot{x}_1 &= x_2 \\ \vdots & \vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -\sum_i k_i \text{sgn}(x_i) \end{cases}.$$

It is easy to check that this vector field is homogeneous of degree  $-1$  w.r.t. to the generalized weight  $(n, \dots, 2, 1)$ . The associated differential inclusion is therefore homogeneous w.r.t.  $(n, \dots, 2, 1)$  of degree  $-1$  as well.

#### IV. SOME RESULTS CONCERNING HOMOGENEOUS DIFFERENTIAL INCLUSIONS

##### A. Globally asymptotically stable DI admits a homogeneous Lyapunov function

The following theorem asserts that a strongly globally asymptotically stable system admits a homogeneous Lyapunov function.

This result is a generalization of the theorem of L. Rosier [1].

*Theorem 4.1:* Let  $F$  be a  $\nu$ -homogeneous set-valued map of degree  $m$ , satisfying the standard assumptions. Then the following statements are equivalent:

- The system (7) is (strongly) GAS.
- For all  $k > \max(-m, 0)$ , there exists a pair  $(V, W)$  of continuous functions, such that:
  - 1)  $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ ,  $V$  is positive definite and homogeneous of degree  $k$ ;
  - 2)  $W \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ ,  $W$  is strictly positive outside the origin and homogeneous of degree  $k + m$ ;
  - 3)  $\max_{v \in F(x)} d_x V v \leq -W(x)$  for all  $x \neq 0$ .

The proof is omitted due to space limitations.

A direct consequence of this result deals with the finite-time stability (FTS) defined below:

*Definition 4.2:* The system (7) is said to be FTS if:

- 1) the system is stable;
- 2) there exists an open neighborhood  $\mathcal{U}$  of the origin such that for all  $x \in \mathcal{U}$ , there exists  $\tau \geq 0$  such that for all  $t \geq \tau$ , we have  $\Psi^t(x) = \{0\}$ .

The settling-time function is then defined for  $x \in \mathcal{U}$  by  $T(x) = \inf\{\tau \geq 0 : \forall t \geq \tau, \Psi^t(x) = \{0\}\}$ .

If the neighborhood  $\mathcal{U}$  can be chosen to be  $\mathbb{R}^n$ , the system is said to be Globally FTS (GFTS).

*Corollary 4.3:* Let  $F$  be a  $\nu$ -homogeneous set-valued map of degree  $m < 0$ , satisfying the standard assumptions. Assume also that  $F$  is GAS. Then  $F$  is GFTS and the settling-time function is continuous at zero and locally bounded.

*Proof:* Being homogeneous and GAS,  $F$  admits a homogeneous Lyapunov pair  $(V, W)$ , as established in the previous theorem. Let us apply Lemma 4.2 of [2] to the continuous functions  $V$  and  $W$ . We get that for all  $x \in \mathbb{R}^n \setminus \{0\}$ , and for all  $v \in F(x)$ :

$$d_x V v \leq -W(x) \leq -C(V(x))^{\frac{k+m}{k}}, \quad (9)$$

where  $C = \min_{\{V=1\}} W$ . Since  $\frac{k+m}{k} < 1$ ,  $V$  converges to zero in a finite time, giving us the finite-time convergence of the system, which is therefore GFTS. Moreover, a direct integration of the inequation (9) gives  $T(x) \leq \frac{kV(x)^{\frac{-m}{k}}}{-mC}$ , where  $T$  denotes the settling-time function. Since  $V$  is continuous,  $T$  is locally bounded and continuous at zero. ■

*Remark 4.4:* It has been shown in [2] that under the assumptions of homogeneity (with negative degree), continuity of the right-hand side and forward unicity of solutions, the settling-time function is continuous. Let us emphasize that these assumptions cannot be imposed in our context, thus the settling-time function is *not* continuous in general. See, for instance, [36] or the following example.

*Example 4.5:* (A counterexample to the second statement of Theorem 1 from [29]) Consider the system defined on  $\mathbb{R}^2$  by:

$$\dot{x} = -(\text{sgn}(x_1) + 2) \frac{x}{|x|}.$$

This system is clearly strongly (uniformly [29]) GFTS and homogeneous of negative degree. A simple computation shows that the settling-time function is:

$$\mathbb{T}(x) = \begin{cases} |x| & x_1 \geq 0 \\ \frac{|x|}{3} & x_1 < 0 \end{cases},$$

which is discontinuous on  $x_1 = 0$ .

### B. Qualitative results on the trajectories of a homogeneous differential inclusion

In this section, we will be interested in the solutions of the differential inclusion (7). We assume in this section that  $F$  satisfies the standard assumptions. Let us start with technical lemmas.

*Lemma 4.6:* Let  $K$  be a compact subset of  $\mathbb{R}^n$ . There exists  $T > 0$  such that any solution of (7) starting in  $K$  can be extended to the whole interval  $[0, T]$  and for all  $t \in [0, T]$  the set  $\Psi^t(K)$  is bounded.

*Proof:* Consider a compact subset  $L$  of  $\mathbb{R}^n$  such that  $K \subset \overset{\circ}{L}$ . Denote  $\delta = d(K, \partial L) > 0$  and  $M = \max\{|v| : v \in F(x), x \in L\}$ . The positive number  $M$  is well-defined since the set  $\{v \in F(x), x \in L\}$  is compact. If  $M = 0$ ,  $\Psi^t(K) = K$  for any  $t \geq 0$ . We assume now that  $M \neq 0$ .

Let  $x$  be a solution of (7) with initial condition in  $K$  and denote  $\tau = \inf\{t \geq 0 : x(t) \in \partial L\}$ . Hence, for all  $0 \leq t \leq \tau$ ,  $x(t) \in L$  and thus:

$$|x(\tau) - x(0)| \leq \int_0^\tau |\dot{x}(u)| du \leq M\tau.$$

Since  $x(\tau) \in \partial L$  and  $x(0) \in K$ , we have  $|x(\tau) - x(0)| \geq \delta$ . Thus  $\tau \geq \delta/M$ . We can therefore extend any solution of (7) starting in  $K$  on the interval  $[0, T]$ , where  $T = \delta/M$ . Moreover, for any  $t \in [0, T]$ , we have  $x(t) \in L$ . ■

*Proposition 4.7:* Let  $K$  be a compact subset of  $\mathbb{R}^n$ . Assume that there exists  $T > 0$  such that every trajectories of (7) starting in  $K$  stay in the compact  $L$  for all  $t \in [0, T]$ . Then for all  $t \in [0, T]$ , the set  $\Psi^t(K)$  is compact.

*Proof:* For all  $t \in [0, T]$  the set  $\Psi^t(K)$  is bounded. Let us show that the set  $\Psi^t(K)$  is then compact. Consider a sequence of points  $(x_n(t)) \in \Psi^t(K)$  with corresponding trajectories  $(x_n) \in \mathcal{S}([0, t], K)$ . This set of trajectories is bounded and equicontinuous, since the derivatives of the trajectories are bounded by  $M$  almost everywhere:  $|x_n(b) - x_n(a)| \leq M(b - a)$ . By the Arzela-Ascoli theorem, this sequence admits a subsequence (we do not relabel) uniformly converging to a continuous function  $x$ . Since all the functions  $x_n$  are  $M$ -Lipchitz, so is  $x$ ; finally  $x$  is absolutely continuous.

Let  $\mathcal{V}$  be a compact convex neighborhood of  $F(x(t))$ . By USC of  $F$ , there exists an open bounded neighborhood  $\mathcal{U}$  of  $x(t)$  such that for all  $y \in \mathcal{U}$ ,  $F(y) \subset \mathcal{V}$ . Since  $x$  is continuous, there exists  $\eta > 0$  such that for all  $\tau \in [t - \eta, t + \eta]$ ,  $x(\tau) \in \mathcal{U}$ . Let us denote  $I = \{x(\tau) : \tau \in [t - \eta, t + \eta]\}$ . The set  $I$  is compact and is a subset of  $\mathcal{U}$ . Set  $\alpha = d(I, \partial \mathcal{U}) > 0$ . Since  $(x_n)$  converges uniformly to  $x$ , there exists  $N > 0$  such that for all  $n \geq N$  and for all  $\tau \in [t - \eta, t + \eta]$ ,  $|x_n(\tau) - x(\tau)| \leq \frac{\alpha}{2}$ . Thus for

all  $n \geq N$  and for all  $\tau \in [t - \eta, t + \eta]$ ,  $x_n(\tau) \in \mathcal{U}$  and then  $\dot{x}_n(t) \in \mathcal{V}$ . Applying now the lemma 4.3 from [37], we get that  $\dot{x}(t) \in \mathcal{V}$ . Being compact and convex,  $F(x(t))$  is equal to the intersection of all its compact and convex neighborhood. Therefore,  $\dot{x}(t) \in F(x(t))$ . Since  $x(0) = \lim x_n(0)$  and  $x_n(0) \in K$  compact,  $x(0) \in K$ . Finally,  $x \in \mathcal{S}([0, T], K)$  and  $x(t) \in \Psi^t(K)$ . We have proved that every sequence in  $\Psi^t(K)$  admits a converging subsequence:  $\Psi^t(K)$  is compact. ■

We can now formulate the generalization of Theorem 6.1 of [2].

*Corollary 4.8:* Suppose that  $K$  is a strongly strictly positively invariant subset (SPI) of  $\mathbb{R}^n$  for the homogeneous system (7). Then the origin is GAS for (7).

*Example 4.9:* Assume we know  $k_1, k_2 > 0$  such that the following system is GAS:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1 \text{sgn}(x_1) - k_2 \text{sgn}(x_2) \end{cases}.$$

As we have seen, this system is homogeneous w.r.t. the weight  $(2, 1)$  of degree  $-1$ . Therefore, there exists a homogeneous Lyapunov pair  $(V, W)$  of degrees  $k > 1$  and  $k - 1$ . We denote  $F_0$  the set valued map associated to this vector field and for  $\alpha \in \mathbb{R}^2$  we denote  $F_\alpha$  the set valued map associated to the vector field:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(k_1 + \alpha_1) \text{sgn}(x_1) - (k_2 + \alpha_2) \text{sgn}(x_2) \end{cases}.$$

We shall prove that the compact set  $K = \{x \in \mathbb{R}^2 : V(x) \leq 1\}$  is SPI for  $F_\alpha$ , for small values of  $\alpha$ . Let  $y \in K$  and  $v \in F_\alpha(y)$ . There exists  $x \in S = \{x \in \mathbb{R}^n : V(x) = 1\}$  and  $s \in \mathbb{R}$  such that  $\Phi^s(x) = y$ . By homogeneity, there also exists  $w \in F_\alpha(x)$  such that  $v = e^{-s} d_x \Phi^s w$ . Therefore:

$$d_y V v = d_{\Phi^s(x)} V e^{-s} d_x \Phi^s w = e^{(k-1)s} d_x V w.$$

Since  $w \in F_\alpha(x)$ , there exist  $\sigma_1, \sigma_2 \in [-1, 1]$  such that  $w = (x_2, -(k_1 + \alpha_1)\sigma_1 - (k_2 + \alpha_2)\sigma_2)^T$ . Let us denote  $\tilde{w} = (x_2, -k_1\sigma_1 - k_2\sigma_2)^T \in F_0(x)$ . We have:

$$\begin{aligned} d_y V v &= e^{(k-1)s} d_x V w \\ &= e^{(k-1)s} [d_x V \tilde{w} + d_x V (w - \tilde{w})] \\ &\leq e^{(k-1)s} \left[ -W(x) + \sup_{x \in S} |d_x V| |w - \tilde{w}| \right] \\ &\leq e^{(k-1)s} \left[ -\inf_{x \in S} W(x) + \sup_{x \in S} |d_x V| \cdot \sum_{i=1}^2 |\alpha_i \sigma_i| \right] \\ &\leq e^{(k-1)s} \left[ -\inf_{x \in S} W(x) + \sup_{x \in S} |d_x V| \cdot \sum_{i=1}^2 |\alpha_i| \right]. \end{aligned}$$

Therefore, if  $|\alpha_1| + |\alpha_2| < \frac{\inf_{x \in S} W(x)}{\sup_{x \in S} |d_x V|}$ ,  $d_y V v < 0$ , which means that  $K$  is strictly positively invariant. The set valued map  $F_\alpha$  being homogeneous, it is hence GAS for  $\alpha$  small enough. Finally, any stabilizing control under the form  $u(x) = -k_1 \text{sgn}(x_1) - k_2 \text{sgn}(x_2)$  is robust w.r.t. small errors on the gains  $k_i$ , like implementation errors.

## V. CONCLUSION

- A geometric definition of homogeneity for DIs is proposed, which is consistent with the regularization procedure of Filipov.
- An extension of the theorem of L. Rosier is given: if a homogeneous DI is strongly globally asymptotically stable and satisfies the standard assumptions for DI (existence of solutions), then there exists a homogeneous proper Lyapunov function.
- An extension of a well known result about FTS is developed for DI: if a homogeneous DI is globally asymptotically stable with negative degree and satisfies the standard assumptions, then the DI is strongly FTS.
- As in the continuous-time case, it is shown that for a homogeneous DI the existence of a SPI compact set is equivalent to global asymptotic stability.

These results are, from our point of view, the main ingredients for a possible route to a complete theory of design and analysis for higher order sliding mode.

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