Estimation and Control of Discrete-Time LPV Systems Using Interval Observers

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***Abstract*—This work is devoted to interval observers design for discrete-time Linear Parameter-Varying (LPV) systems under assumption that the vector of scheduling parameters is not available for measurements. Two scenarios are considered: a pure estimation and an output stabilizing feedback design. Stability conditions are expressed in terms of linear matrix inequalities. The efficiency of the proposed approach is demonstrated through computer simulations.**

I. Introduction

Many nonlinear system models can be represented by (or converted into) an LPV form. The main advantage is that a partial linearity of LPV models allows one to apply several methods developed for linear systems [1], [2], [3], [4]. The problem of state estimation for nonlinear systems is rather challenging and finds many applications [5], [6], [7]. Frequently, in the nonlinear case, observer or controller design are based on transformation of the system to a canonical form, then the design of such a transformation can be an obstruction in practice. This limitation motivates the application of LPV systems framework.

In some situations due to the presence of uncertainties (unknown parameters or/and disturbances) the design of a conventional estimator, converging in the noise-free case to the exact value of the state, is complicated. Especially it is the case when the vector of scheduling parameters of an LPV system is not available for measurements (partially or completely). However, even in this situation an interval estimation may still remain feasible. By interval estimation we understand an observer that, using input-output information and the bounds of the model uncertainties, evaluates the set of admissible values (interval) for the state at each instant of time.

There are several approaches to design interval observers [8], [9], [10], [11]. This paper continues the framework of interval observer design based on monotone systems theory [10], [11], [12], [13], [14]. One of the most restrictive assumptions for an interval observer design deals with cooperativity of the interval estimation error dynamics, which was recently relaxed in [15], [13], [16]. In those studies, it has been shown that under some mild conditions applying a similarity transformation, a Hurwitz matrix could be transformed to a Hurwitz and Metzler one (cooperative). In order to design an interval observer for systems with non-constant matrices dependent on measurable input-output signals and time, an extension of the result from [13] has been presented in [17], which allows one to calculate a constant similarity transformation matrix representing a given interval of matrices into an interval of Metzler matrices. Similar results for discrete-time systems have been recently obtained in [18], [19].

The problem becomes more challenging if a control design is required. There exists a few methods, which are capable to stabilize an uncertain nonlinear system using only output measurements, and the LPV system approach is one of the most popular among them [20], [21]. Interval observers can be used for the stabilization of a continuous-time LPV system (or a class of nonlinear uncertain systems) under assumption that the vector of scheduling parameters is not measured [22].

The first objective of this work is to extend the results from [19] on interval observer design for discrete-time LPV systems with an unmeasurable vector of scheduling parameters. The second objective is to design a stabilizing control based on interval observers as in [22]. The paper is organized as follows. Some basic facts from the theory of interval estimation are given in Section 2. Problem formulation and some preliminaries are presented in Section 3. The main result is described in Section 4, where an interval observer is designed and next it is shown how it can be used for stabilization. Examples of computer simulations are presented in Section 5.

II. Preliminaries

The real and integer numbers are denoted by $\mathbb{R}$ and $\mathbb{Z}$ respectively, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. Euclidean norm of a vector $x \in \mathbb{R}^n$ will be denoted by

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$|x|$, and for a measurable and locally essentially bounded input $u:\mathbb{Z}\to\mathbb{R}$ (we will use the convention $u(t)=u_t$ for a $t\in\mathbb{Z}_+$) the symbol $||u||_{[t_0,t_1)}$ denotes its L_∞ norm $||u||_{[t_0,t_1)}=\sup\{|u_t|,t\in[t_0,t_1)\}$, $||u||=||u||_{[0,+\infty)}$. We will denote by $\mathcal{L}_\infty$ the set of all inputs u with the property $||u||<\infty$. Denote the sequence of integers $1,...,k$ by $\overline{1,k}$. The symbols I_n, $E_{n\times m}$ and E_p denote the identity matrix with dimension $n\times n$, the matrix with all elements equal 1 with dimensions $n\times m$ and $p\times 1$ respectively. For a symmetric matrix $A\in\mathbb{R}^{n\times n}$, we denote by $\lambda(A)$ the vector of its eigenvalues, $\lambda_{\max}(A)=\max\lambda(A)$, $\lambda_{\min}(A)=\min\lambda(A)$ and by $||A||_2=\sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$ the induced matrix norm.

A. Interval analysis

For two vectors $x_1,x_2\in\mathbb{R}^n$ or matrices $A_1,A_2\in\mathbb{R}^{n\times n}$, the relations $x_1\leq x_2$ and $A_1\leq A_2$ are understood elementwise. The relation $P\prec 0$ ($P\succeq 0$) means that the matrix $P\in\mathbb{R}^{n\times n}$ is negative (positive semi-) definite. Given a matrix $A\in\mathbb{R}^{m\times n}$, define $A^+=\max\{0,A\}$, $A^-=A^+-A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A|=A^++A^-$.

Lemma 1. *Let $\underline{A}\leq A\leq\overline{A}$ for some $\underline{A},\overline{A},A\in\mathbb{R}^{n\times n}$ and $\underline{x}\leq x\leq\overline{x}$ for $\underline{x},\overline{x},x\in\mathbb{R}^n$, then*

$$\begin{aligned}&\underline{A}^+\underline{x}^+-\overline{A}^+\underline{x}^--\underline{A}^-\overline{x}^++\overline{A}^-\overline{x}^-\leq Ax\\&\leq\overline{A}^+\overline{x}^+-\underline{A}^+\overline{x}^--\overline{A}^-\underline{x}^++\underline{A}^-\underline{x}^-.\end{aligned}\quad(1)$$

Proof: By definition $Ax=(A^+-A^-)(x^+-x^-)=A^+x^+-A^+x^--A^-x^++A^-x^-$, where all terms are elementwise nonnegative, which gives the required relations. ■

B. Cooperative discrete-time linear systems

A matrix $A\in\mathbb{R}^{n\times n}$ is called Schur stable if all its eigenvalues have the norm less than one, it is called nonnegative if all its elements are nonnegative (i.e $A\geq 0$), and it is called Metzler if all its off-diagonal elements are nonnegative. Any solution of the system

$$x_{t+1}=Ax_t+\omega_t,\ \omega:\mathbb{Z}_+\to\mathbb{R}_+^n,\ t\in\mathbb{Z}_+,$$

with $x_t\in\mathbb{R}^n$ and a nonnegative matrix $A\in\mathbb{R}_+^{n\times n}$, is elementwise nonnegative for all $t\geq 0$ provided that $x(0)\geq 0$ [23]. Such a system is called cooperative (monotone) or nonnegative [23].

Lemma 2. [24] *A matrix $A\in\mathbb{R}_+^{n\times n}$ is Schur stable iff there exists a diagonal matrix $D\in\mathbb{R}_+^{n\times n}$ such that $A^TDA-D\prec 0$.*

In the sequel we are interested in a Luenberger-like observer design with the gain L such that the matrix $A-LC$ (the closed loop matrix of the estimation error dynamics) would be Schur stable and nonnegative. Usually it is not possible to find such a matrix L. However a change of variables $z(t)=Sx(t)$ with a nonsingular matrix S can be proposed such that, in the new coordinates, the matrix $S(A-LC)S^{-1}$ would satisfy the required properties. An idea how to design such a matrix S is given in the lemma below.

Lemma 3. [13] *Given the matrices $A\in\mathbb{R}^{n\times n}$, $R\in\mathbb{R}^{n\times n}$ and $C\in\mathbb{R}^{p\times n}$. If there exists a matrix $L\in\mathbb{R}^{n\times p}$ such that $\lambda(A-LC)=\lambda(R)$, and there exist vectors $\varrho_1\in\mathbb{R}^{1\times n}$, $\varrho_2\in\mathbb{R}^{1\times n}$ such that the pairs $(A-LC,\varrho_1)$ and (R,ϱ_2) are observable, then there is a nonsingular $S\in\mathbb{R}^{n\times n}$ such that $R=S(A-LC)S^{-1}$.*

This result was used in [13] to design interval observers for continuous-time LTI systems with a Metzler matrix R. The main difficulty is to prove existence of a *real* and *nonsingular* matrix S, and to provide a constructive approach of its calculation. In the work [19], Lemma 3 has been also applied to a nonnegative matrix R.

III. Problem statement

Consider an LPV system described by:

$$\begin{aligned}x_{t+1}&=[A_0+\Delta A(\rho_t)]x_t+Bu_t+d_t,\\y_t&=Cx_t+v_t,\ t\in\mathbb{Z}_+,\end{aligned}\quad(2)$$

where $x_t\in\mathbb{R}^n$ is the state, $y_t\in\mathbb{R}^p$ is the output available for measurements, $u_t\in\mathbb{R}^m$ is the control, $\rho_t\in\Pi\subset\mathbb{R}^r$ is the vector of scheduling parameters with a known Π, $\rho\in\mathcal{L}_\infty^r$. The values of the scheduling vector ρ are not available for measurements, and only the set of admissible values Π is given. The matrices $A_0\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times n}$ and $C\in\mathbb{R}^{p\times n}$ are known, the matrix function $\Delta A:\Pi\to\mathbb{R}^{n\times n}$ is piecewise continuous and also known. The signals $d\in\mathcal{L}_\infty^n$ and $v\in\mathcal{L}_\infty^p$ are the exogenous disturbance and measurement noise respectively, the exact current values of d_t and v_t are not available. For brevity of presentation we will use the following assumptions in this work.

Assumption 1. *$\underline{d}_t\leq d_t\leq\overline{d}_t$ and $|v_t|\leq V$ for all $t\in\mathbb{Z}_+$ and for some known $\underline{d},\overline{d}\in\mathcal{L}_\infty^n$ and $V\in\mathbb{R}_+$.*

Assumption 2. *$\underline{\Delta A}\leq\Delta A(\rho)\leq\overline{\Delta A}$ for all $\rho\in\Pi$ and some known $\underline{\Delta A},\overline{\Delta A}\in\mathbb{R}^{n\times n}$.*

Therefore, it is assumed that the measurement noise v_t has an upper bound V and the input d_t belongs to a known bounded interval $[\underline{d}_t,\overline{d}_t]$ for all $t\in\mathbb{Z}_+$. It is also assumed that the matrix $\Delta A(\rho)$ belongs to the interval $[\underline{\Delta A},\overline{\Delta A}]$ for all $t\in\mathbb{Z}_+$, which is easy to compute for a given set Π and a known function $\Delta A:\Pi\to\mathbb{R}^{n\times n}$ (in a polytopic case, for example).

An interval observer for a class of continuous-time LPV systems has been proposed in [22] in the framework of stabilization of LPV systems. In that work the stability of interval observers was ensured by a proper choice of control input providing the uncertain system stabilization. For the case of a measured vector ρ_t in (2), an interval observer was proposed in [19] using the result of Lemma 3. Let us consider how to extend these results to a discrete-time LPV system (2) with unmeasured ρ.

Before introduction of the interval observer equations note that for a matrix $L \in \mathbb{R}^{n\times p}$, the system (2) can be rewritten as follows:

$$\begin{aligned} x_{t+1} &= [A_0 - LC]x_t + \Delta A(\rho_t)x_t \\ &\quad + L[y_t - v_t] + Bu_t + d_t, \end{aligned}$$

and according to Lemma 1 and Assumption 2, we have for all $\rho \in \Pi$:

$$\begin{aligned} &\underline{\Delta A}^+ \underline{x}_t^+ - \overline{\Delta A}^+ \underline{x}_t^- - \underline{\Delta A}^- \overline{x}_t^+ + \overline{\Delta A}^- \overline{x}_t^- \le \Delta A(\rho_t)x_t \\ &\le \overline{\Delta A}^+ \overline{x}_t^+ - \underline{\Delta A}^+ \overline{x}_t^- - \overline{\Delta A}^- \underline{x}_t^+ + \underline{\Delta A}^- \underline{x}_t^- \qquad (3) \end{aligned}$$

provided that $\underline{x}_t \le x_t \le \overline{x}_t$ for any $\underline{x}_t, \overline{x}_t, x_t \in \mathbb{R}^n$, $\rho_t \in \Pi$ and $t \in \mathbb{Z}_+$.

The design of an interval observer can be considerably simplified under the assumption that $x_t \in \mathbb{R}^n_+$ and $\underline{d}_t \in \mathbb{R}^n_+$ for all $t \in \mathbb{Z}_+$, i.e. the system (2) is cooperative (nonnegative), but in this work we will not impose such an assumption.

IV. Main results

The interval observer equations for the system (2) take the form:

$$\begin{aligned} \underline{x}_{t+1} &= [A_0 - \underline{L}C]\underline{x}_t + Bu_t + [\underline{\Delta A}^+ \underline{x}_t^+ - \overline{\Delta A}^+ \underline{x}_t^- \\ &\quad - \underline{\Delta A}^- \overline{x}_t^+ + \overline{\Delta A}^- \overline{x}_t^-] + \underline{L}y_t - |\underline{L}|VE_p + \underline{d}_t, \\ \overline{x}_{t+1} &= [A_0 - \overline{L}C]\overline{x}_t + Bu_t + [\overline{\Delta A}^+ \overline{x}_t^+ - \underline{\Delta A}^+ \overline{x}_t^- \qquad (4) \\ &\quad - \overline{\Delta A}^- \underline{x}_t^+ + \underline{\Delta A}^- \underline{x}_t^-] + \overline{L}y_t + |\overline{L}|VE_p + \overline{d}_t, \end{aligned}$$

where $\underline{x}_t$, $\overline{x}_t$ are the lower ad upper interval estimates of x_t, $\underline{L} \in \mathbb{R}^{n\times p}$ and $\overline{L} \in \mathbb{R}^{n\times p}$ are the observer gains to be designed. Note that due to the presence of $\underline{x}_t^+, \underline{x}_t^-, \overline{x}_t^+$ and $\overline{x}_t^-$, the interval observer (4) is a globally Lipschitz *nonlinear* system.

Next, two cases are considered in this section. *Firstly*, assuming that the state x_t is bounded and the control signal u_t is given, the goal is to design two gains $\underline{L}$ and $\overline{L}$ such that (4) ensures the interval estimation of x_t (i.e. $\underline{x}_t \le x_t \le \overline{x}_t$ for all $t \in \mathbb{Z}_+$) and boundedness of $\underline{x}_t$, $\overline{x}_t$. *Secondly*, the interval observer (4) with the prior designed gains $\underline{L}$, $\overline{L}$ is used to compute a control law $u_t = U(\underline{x}_t, \overline{x}_t, y_t)$ guaranteeing an interval estimation of the state x_t and its boundedness together with $\underline{x}_t$, $\overline{x}_t$.

A. Interval estimation

The conditions, which have to be imposed on the gains $\underline{L}$, $\overline{L}$ in order to ensure an interval estimation of x_t and the boundedness of $\underline{x}_t$, $\overline{x}_t$, are formulated in the following theorem.

Theorem 1. *Let assumptions 1, 2 be satisfied,* $x \in \mathcal{L}^n_\infty$, $u \in \mathcal{L}^m_\infty$ *and*

$$A_0 - \underline{L}C,\ A_0 - \overline{L}C \in \mathbb{R}^{n\times n}_+. \qquad (5)$$

Then, the relations

$$\underline{x}_t \le x_t \le \overline{x}_t \quad \forall t \in \mathbb{Z}_+ \qquad (6)$$

are satisfied provided that $\underline{x}_0 \le x_0 \le \overline{x}_0$. *In addition, if there exist a diagonal matrix* $P \in \mathbb{R}^{2n\times 2n}$, $P \succ 0$ *(i.e.* $P > 0$*), a matrix* $Q \in \mathbb{R}^{2n\times 2n}$, $Q = Q^T \succ 0$ *and constants* $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\gamma > 0$ *such that the following matrix inequality is verified*

$$\begin{aligned} \Phi &= \begin{bmatrix} \Psi & G^T P \\ PG & (1+\epsilon_2)P - \gamma I_{2n} \end{bmatrix} \preceq 0, \qquad (7) \\ \Psi &= (1+\epsilon_1)G^T PG - P + Q + \gamma\eta^2 I_{2n}, \end{aligned}$$

where $\eta = 2(||\underline{\Delta A}^+ - \overline{\Delta A}^+||_2 + ||\underline{\Delta A}^-||_2 + ||\overline{\Delta A}^-||_2)$ *and*

$$G = \begin{bmatrix} A_0 - \underline{L}C + \underline{\Delta A}^+ & 0 \\ 0 & A_0 - \overline{L}C + \overline{\Delta A}^+ \end{bmatrix},$$

then $\underline{x}, \overline{x} \in \mathcal{L}^n_\infty$.

Proof: Consider the dynamics of interval estimation errors $\underline{e}_t = x_t - \underline{x}_t$ and $\overline{e}_t = \overline{x}_t - x_t$:

$$\begin{aligned} \underline{e}_{t+1} &= [A_0 - \underline{L}C]\underline{e}_t + \sum_{i=1}^{3} \underline{w}_t^i, \\ \overline{e}_{t+1} &= [A_0 - \overline{L}C]\overline{e}_t + \sum_{i=1}^{3} \overline{w}_i^t, \end{aligned}$$

where

$$\begin{aligned} \underline{w}_t^1 &= \Delta A(\rho_t)x_t - \underline{\Delta A}^+ \underline{x}_t^+ \\ &\quad - \overline{\Delta A}^+ \underline{x}_t^- - \underline{\Delta A}^- \overline{x}_t^+ + \overline{\Delta A}^- \overline{x}_t^-, \\ \underline{w}_t^2 &= |\underline{L}|VE_p - \underline{L}v_t,\ \underline{w}_t^3 = d_t - \underline{d}_t; \\ \overline{w}_t^1 &= \overline{\Delta A}^+ \overline{x}_t^+ - \underline{\Delta A}^+ \overline{x}_t^- \\ &\quad - \overline{\Delta A}^- \underline{x}_t^+ + \underline{\Delta A}^- \underline{x}_t^- - \Delta A(\rho_t)x_t, \\ \overline{w}_t^2 &= \overline{L}v_t + |\overline{L}|VE_p,\ \overline{w}_t^3 = \overline{d}_t - d_t. \end{aligned}$$

Let the gains $\underline{L}$, $\overline{L}$ be designed in order to verify (5) and suppose that $\underline{x}_0 \le x_0 \le \overline{x}_0$, then the dynamics for $\underline{e}_t$, $\overline{e}_t$ is cooperative and $\underline{e}_0, \overline{e}_0 \in \mathbb{R}_+$, thus if $\underline{w}_t^i \ge 0$, $\overline{w}_t^i \ge 0$ for $i = \overline{1,3}$ and all $t \in \mathbb{Z}_+$, then the relations (6) are satisfied. The inputs $\underline{w}_t^i$, $\overline{w}_t^i$, $i = 2,3$ are nonnegative for all $t \in \mathbb{Z}_+$ due to Assumption 1. To prove that $\underline{w}_t^1$ and $\overline{w}_t^1$ remain nonnegative while the relation (6) is satisfied let us recall that by Assumption 2 the relations (3) are satisfied whereas the relation (6) is true, (3) implies that $\underline{w}_t^1 \ge 0$ and $\overline{w}_t^1 \ge 0$ for all $t \in \mathbb{Z}_+$. However, the relation (6) is valid at time $t = 0$ and it is preserved for all $t \in \mathbb{Z}_+$ by induction and cooperativity of dynamics of estimation errors $\underline{e}_t$, $\overline{e}_t$, therefore (6) and (3) are valid.

Let us show that the variables $\overline{x}_t$ and $\underline{x}_t$ stay bounded for all $t \in \mathbb{Z}_+$ in (2), (4). For this purpose let us rewrite the equations (4) as follows:

$$\begin{aligned} \underline{x}_{t+1} &= [A_0 - \underline{L}C + \underline{\Delta A}^+]\underline{x}_t + \underline{f}(\underline{x}_t, \overline{x}_t) + \underline{\delta}_t, \qquad (8) \\ \overline{x}_{t+1} &= [A_0 - \overline{L}C + \overline{\Delta A}^+]\overline{x}_t + \overline{f}(\underline{x}_t, \overline{x}_t) + \overline{\delta}_t, \end{aligned}$$

where

$$\underline{\delta}_t = \underline{L}y_t - |\underline{L}|VE_p + Bu_t + \underline{d}_t,$$
$$\overline{\delta}_t = \overline{L}y_t + |\overline{L}|VE_p + Bu_t + \overline{d}_t,$$
$$\underline{f}(\underline{x},\overline{x}) = (\underline{\Delta A}^+ - \overline{\Delta A}^+)\underline{x}^- - \underline{\Delta A}^-\overline{x}^+ + \overline{\Delta A}^-\overline{x}^-,$$
$$\overline{f}(\underline{x},\overline{x}) = (\overline{\Delta A}^+ - \underline{\Delta A}^+)\overline{x}^- - \overline{\Delta A}^-\underline{x}^+ + \underline{\Delta A}^-\underline{x}^-.$$

The system (8) is not cooperative and the variables $\underline{x},\overline{x}$ are interrelated, however its linear part is nonnegative (the matrices $A_0 - \underline{L}C + \underline{\Delta A}^+$ and $A_0 - \overline{L}C + \overline{\Delta A}^+$ are nonnegative since $A_0 - \underline{L}C$, $A_0 - \overline{L}C$, $\underline{\Delta A}^+$ and $\overline{\Delta A}^+$ are nonnegative ones), and the inputs $\underline{\delta}_t$ and $\overline{\delta}_t$ are bounded by Assumption 1 and the facts that $x \in \mathcal{L}_\infty^n$, $u \in \mathcal{L}_\infty^m$. The boundedness of $\underline{x}_t, \overline{x}_t$ is predefined by properties of the linear part and the functions $\underline{f}$, $\overline{f}$. Clearly $\underline{f}$ and $\overline{f}$ are globally Lipschitz:

$$\begin{aligned}
|\underline{f}(\underline{x},\overline{x})| &\leq ||\underline{\Delta A}^+ - \overline{\Delta A}^+||_2|\underline{x}| \\
&\quad +(||\underline{\Delta A}^-||_2 + ||\overline{\Delta A}^-||_2)|\overline{x}|, \\
|\overline{f}(\underline{x},\overline{x})| &\leq ||\overline{\Delta A}^+ - \underline{\Delta A}^+||_2|\overline{x}| \\
&\quad +(||\overline{\Delta A}^-||_2 + ||\underline{\Delta A}^-||_2)|\underline{x}|.
\end{aligned}$$

To prove boundedness of the solutions of the observer (4), introduce the system

$$\xi_{t+1} = G\xi_t + \phi(\xi_t) + \delta_t,$$

where

$$\xi_t = \begin{bmatrix} \underline{x}_t \\ \overline{x}_t \end{bmatrix},\ \phi(\xi_t) = \begin{bmatrix} \underline{f}(\underline{x}_t,\overline{x}_t) \\ \overline{f}(\underline{x}_t,\overline{x}_t) \end{bmatrix},\ \delta_t = \begin{bmatrix} \underline{\delta}_t \\ \overline{\delta}_t \end{bmatrix},$$
$$|\phi(\xi_t)| \leq \eta|\xi_t|$$

and $\delta \in \mathcal{L}_\infty^{2n}$. Let us consider a Lyapunov function $V_t = \xi_t^{\mathrm{T}}P\xi_t$, whose increment takes the form:

$$\begin{aligned}
V_{t+1} - V_t &= \xi_t^{\mathrm{T}}[G^{\mathrm{T}}PG - P]\xi_t + \xi_t^{\mathrm{T}}G^{\mathrm{T}}P\phi(\xi_t) \\
&\quad +\phi^{\mathrm{T}}(\xi_t)PG\xi_t + \phi^{\mathrm{T}}(\xi_t)P\phi(\xi_t) \\
&\quad +2\xi_t^{\mathrm{T}}G^{\mathrm{T}}P\delta_t + 2\delta_t^{\mathrm{T}}P\phi(\xi_t) + \delta_t^{\mathrm{T}}P\delta_t \\
&\leq \xi_t^{\mathrm{T}}[(1+\epsilon_1)G^{\mathrm{T}}PG - P]\xi_t + \xi_t^{\mathrm{T}}G^{\mathrm{T}}P\phi(\xi_t) \\
&\quad +\phi^{\mathrm{T}}(\xi_t)PG\xi_t + (1+\epsilon_2)\phi^{\mathrm{T}}(\xi_t)P\phi(\xi_t) \quad (9)\\
&\quad +(1+\epsilon_1^{-1}+\epsilon_2^{-1})\delta_t^{\mathrm{T}}P\delta_t + \gamma\eta^2\xi_t^{\mathrm{T}}\xi_t \\
&\quad -\gamma\phi^{\mathrm{T}}(\xi_t)\phi(\xi_t) + \xi_t^{\mathrm{T}}Q\xi_t - \xi_t^{\mathrm{T}}Q\xi_t \\
&\leq \begin{bmatrix} \xi_t \\ \phi(\xi_t) \end{bmatrix}^{\mathrm{T}} \Phi \begin{bmatrix} \xi_t \\ \phi(\xi_t) \end{bmatrix} - \xi_t^{\mathrm{T}}Q\xi_t \\
&\quad +(1+\epsilon_1^{-1}+\epsilon_2^{-1})\delta_t^{\mathrm{T}}P\delta_t, \\
&\leq -\xi_t^{\mathrm{T}}Q\xi_t + (1+\epsilon_1^{-1}+\epsilon_2^{-1})\delta_t^{\mathrm{T}}P\delta_t.
\end{aligned}$$

Then $\overline{x}_t$ and $\underline{x}_t$ stay bounded for all $t \in \mathbb{Z}_+$. ■

Optimizing values of the constants ϵ_1, ϵ_2 and the matrices Q and P, it is possible to regulate the accuracy of interval estimation since the signal δ_t represents the influence of uncertainty of the model (2), and the gain of the transfer from δ_t to ξ_t characterizes the width of the interval $[\underline{x}_t, \overline{x}_t]$. This optimization problem is skipped for brevity of presentation.

Due to diagonal structure of the matrix P, the conditions (5), (7) from Theorem 1 can be reformulated in terms of LMIs with respect to $\underline{L}$, $\overline{L}$ and P. Indeed, $G = D - \Lambda\Upsilon$ where

$$D = \begin{bmatrix} A_0 + \underline{\Delta A}^+ & 0 \\ 0 & A_0 + \overline{\Delta A}^+ \end{bmatrix},\ \Lambda = \begin{bmatrix} \underline{L} & 0 \\ 0 & \overline{L} \end{bmatrix},$$
$$\Upsilon = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}.$$

We can rewrite the condition (7) as follows:

$$\begin{bmatrix} P - Q - \gamma\eta^2 I_{2n} & 0 \\ 0 & \gamma I_{2n} - (1+\epsilon_2-(1+\epsilon_1)^{-1})P \end{bmatrix}$$
$$-\begin{bmatrix} G^{\mathrm{T}}P \\ (1+\epsilon_1)^{-1}P \end{bmatrix}(1+\epsilon_1)P^{-1}\begin{bmatrix} G^{\mathrm{T}}P \\ (1+\epsilon_1)^{-1}P \end{bmatrix}^{\mathrm{T}} \succeq 0.$$

Then introducing a new variable $W = P\Lambda$ ($PG = PD - W\Upsilon$) and using the Schur complement we derive an LMI that is equivalent to (7):

$$\begin{bmatrix} \frac{P}{1+\epsilon_1} & PD - W\Upsilon & \frac{P}{1+\epsilon_1} \\ (PD - W\Upsilon)^{\mathrm{T}} & P - Q - \gamma\eta^2 I_{2n} & 0 \\ \frac{P}{1+\epsilon_1} & 0 & \gamma I_{2n} - \varepsilon P \end{bmatrix} \succeq 0, \quad (10)$$
$$P \succ 0,\ \varepsilon = 1 + \epsilon_2 - (1+\epsilon_1)^{-1},$$

which has to be verified with a linear constraint corresponding to (5):

$$P\begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix} - W\Upsilon \geq 0. \quad (11)$$

The matrix variable P has to be declared diagonal, and W has to be declared block-diagonal, then these linear inequalities can be solved using a numerical routine (YALMIP toolbox of MATLAB [25], for instance, as it is done in the examples below).

Remark 1. The requirement that the matrices $A_0 - \underline{L}C$, $A_0 - \overline{L}C$ have to be nonnegative can be relaxed by means of a change of coordinates $z = Tx$ with a nonsingular matrix T such that the matrices $T(A_0 - \underline{L}C)T^{-1}$, $T(A_0 - \overline{L}C)T^{-1}$ are nonnegative. The matrix T can be found using the results of lemmas 3 (looking for $\underline{L} = \overline{L} = L$). This extension is omitted for brevity of presentation.

B. Interval estimation and stabilization

In this subsection the interval observer (4) is used in the design of control law ensuring the stabilization of (2). Thus the assumption on boundedness of $x \in \mathcal{L}_\infty^n$ or $u \in \mathcal{L}_\infty^m$ is no more needed, and only the relations (6) have to be ensured by a proper choice of the gains $\underline{L}$, $\overline{L}$. The control law to be designed should stabilize the interval observer, which due to (6) leads to stabilization of (2).

The main idea of the control synthesis is borrowed from [22]. According to Theorem 1, if in the observer (4) the gains $\underline{L}$, $\overline{L}$ are computed such that the matrices $A_0 - \underline{L}C$ and $A_0 - \overline{L}C$ are nonnegative, then the relations (6) are satisfied for any $u_t \in \mathbb{R}^m$. Therefore, ensuring stabilization of $\overline{x}_t$, $\underline{x}_t$ we provide a similar property for x_t, i.e. the problem of design of an output feedback for stabilization of uncertain system

(2) is replaced by the problem of a state feedback design for a completely known system (4). The only shortcoming, which we meet in this way, is that the dimension of the system (4) is two times larger than the dimension of (2), but the control dimension is not changed.

Theorem 2. *Let assumptions 1, 2 be satisfied with (5), then the relations (6) are true provided that* $\underline{x}_0 \leq x_0 \leq \overline{x}_0$. *If*

$$u_t = \underline{K}\underline{x}_t + \overline{K}\overline{x}_t, \tag{12}$$

where the matrices $\underline{K} \in \mathbb{R}^{m\times n}$, $\overline{K} \in \mathbb{R}^{m\times n}$ *are selected such that there exist a matrix* $P \in \mathbb{R}^{2n\times 2n}$, $P = P^T \succ 0$, *a matrix* $Q \in \mathbb{R}^{2n\times 2n}$, $Q = Q^T \succ 0$ *and constants* $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\gamma > 0$ *such that the matrix inequality (7) is verified, where* $\eta = 2(||\underline{\Delta A}^+ - \overline{\Delta A}^+||_2 + ||\overline{\Delta A}^- - \underline{\Delta A}^-||_2 + \max\{||\underline{L}C||_2, ||\overline{L}C||_2\})$ *and*

$$G = \begin{bmatrix} A_0 - \underline{L}C + \underline{\Delta A}^+ + B\underline{K} & B\overline{K} - \underline{\Delta A}^- \\ B\underline{K} - \overline{\Delta A}^- & A_0 - \overline{L}C + \overline{\Delta A}^+ + B\overline{K} \end{bmatrix},$$

then $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$ *and*

$$|x_t| \leq \left|\left| \begin{bmatrix} \underline{x}_t \\ \overline{x}_t \end{bmatrix} \right|\right| \leq \frac{1}{\sqrt{\lambda_{\min}(P)}}(\sqrt{\lambda_{\max}(P)}\alpha^{0.5t}(|\underline{x}_0| + |\overline{x}_0|)$$
$$+\sqrt{\frac{\beta}{1-\alpha}}[||\underline{d}|| + ||\overline{d}|| + 2(||\underline{L}||_2 + ||\overline{L}||_2)V]),$$

where $\alpha = 1 - \lambda_{\min}(Q)/\lambda_{\max}(P)$ *and* $\beta = (1 + \epsilon_1^{-1} + \epsilon_2^{-1})\lambda_{\max}(P)$.

This proof is omitted due to space limitations.

Remark 2. Note that if there is no measurement noise ($V = 0$) and no additive disturbance ($\underline{d}_t = \overline{d}_t = 0$ for all $t \in \mathbb{Z}_+$), the interval observer-based control (4), (12) provides global asymptotic stability property for an LPV system (2). In this case the uncertainty is presented in the term $\Delta A(\rho_t)x_t$.

In this subsection, the conditions for $\underline{L}$, $\overline{L}$ are given by (5), and that for the control gains $\underline{K}$, $\overline{K}$ by (7), thus the conditions for the observer gains $\underline{L}$, $\overline{L}$ and the controller gains $\underline{K}$, $\overline{K}$ are not independent, but they can be solved consequently. Note that in order to simplify a solution of (7) with respect to $\underline{K}$, $\overline{K}$ we can add some additional constraints to (5) for the design of $\underline{L}$, $\overline{L}$. For example, we can ask for Schur stability of $A_0 - \underline{L}C + \underline{\Delta A}^+$, $A_0 - \overline{L}C + \overline{\Delta A}^+$ or the matrix G under substitution $\underline{K} = \overline{K} = 0$.

In particular, the following LMIs can be formulated to verify an extended condition (5) for the gains $\underline{L}$, $\overline{L}$:

$$\begin{bmatrix} \underline{P} & \underline{P}(A_0 + \underline{\Delta A}^+) - \underline{W}C \\ (A_0 + \underline{\Delta A}^+)^{\mathrm{T}}\underline{P} - C^{\mathrm{T}}\underline{W}^{\mathrm{T}} & \underline{P} \end{bmatrix} \succ 0,$$
$$\begin{bmatrix} \overline{P} & \overline{P}(A_0 + \overline{\Delta A}^+) - \overline{W}C \\ (A_0 + \overline{\Delta A}^+)^{\mathrm{T}}\overline{P} - C^{\mathrm{T}}\overline{W}^{\mathrm{T}} & \overline{P} \end{bmatrix} \succ 0, \tag{13}$$
$$\underline{P}A_0 - \underline{W}C \geq 0,\ \overline{P}A_0 - \overline{W}C \geq 0,\ \underline{P} \succ 0,\ \overline{P} \succ 0,$$

which have to be solved with respect to diagonal matrices $\underline{P} \in \mathbb{R}_+^{n\times n}$, $\overline{P} \in \mathbb{R}_+^{n\times n}$ and some matrix variables $\underline{W} \in \mathbb{R}^{n\times p}$, $\overline{W} \in \mathbb{R}^{n\times p}$, then $\underline{L} = \underline{P}^{-1}\underline{W}$, $\overline{L} = \overline{P}^{-1}\overline{W}$. These LMIs imply (5) and Schur stability of $A_0 - \underline{L}C + \underline{\Delta A}^+$, $A_0 - \overline{L}C + \overline{\Delta A}^+$. In order to rewrite the inequality (7) in a form suitable for numerical solution, let us define

$$D = \begin{bmatrix} A_0 - \underline{L}C + \underline{\Delta A}^+ & -\underline{\Delta A}^- \\ -\overline{\Delta A}^- & A_0 - \overline{L}C + \overline{\Delta A}^+ \end{bmatrix},$$
$$\Xi = \begin{bmatrix} B \\ B \end{bmatrix},\ K = [\underline{K}\ \overline{K}],$$

then $G = D + \Xi K$. Using Schur complement (similarly as in the previous subsection) we obtain

$$\begin{bmatrix} \frac{P}{1+\epsilon_1} & PG & \frac{P}{1+\epsilon_1} \\ G^{\mathrm{T}}P & P - Q - \gamma\eta^2 I_{2n} & 0 \\ \frac{P}{1+\epsilon_1} & 0 & \gamma I_{2n} - \varepsilon P \end{bmatrix} \succeq 0,$$
$$P = P^{\mathrm{T}} \succ 0,\ \varepsilon = 1 + \epsilon_2 - (1+\epsilon_1)^{-1},$$

next multiplying this inequality from the left and from the right by the matrix $\mathrm{diag}[P^{-1}, P^{-1}, P^{-1}]$ we get for $S = P^{-1}$:

$$\begin{bmatrix} \frac{S}{1+\epsilon_1} & GS & \frac{S}{1+\epsilon_1} \\ SG^{\mathrm{T}} & S - Z - \gamma\eta^2 S^2 & 0 \\ \frac{S}{1+\epsilon_1} & 0 & \gamma S^2 - \varepsilon S \end{bmatrix} \succeq 0,$$
$$S = S^{\mathrm{T}} \succ 0,\ Z = Z^{\mathrm{T}} \succ 0,\ \varepsilon = 1 + \epsilon_2 - (1+\epsilon_1)^{-1},$$

where $Z = S^{-1}QS \in \mathbb{R}^{2n\times 2n}$ and $S \in \mathbb{R}^{2n\times 2n}$ are new variables. Applying again Schur complement to the term $-\gamma\eta^2 S^2$ we can rewrite the last LMIs as follows for some $\rho > 0$ and a new variable $M \in \mathbb{R}^{m\times 2n}$:

$$\begin{bmatrix} \gamma^{-1}\eta^{-2}I_{2n} & 0 & S & 0 \\ 0 & \frac{S}{1+\epsilon_1} & DS + \Xi M & \frac{S}{1+\epsilon_1} \\ S & SD^{\mathrm{T}} + M^{\mathrm{T}}\Xi^{\mathrm{T}} & S - Z & 0 \\ 0 & \frac{S}{1+\epsilon_1} & 0 & (\gamma\rho - \varepsilon)S \end{bmatrix} \succeq 0, \tag{14}$$
$$S = S^{\mathrm{T}} \succeq \rho I_{2n},\ Z = Z^{\mathrm{T}} \succ 0,\ \varepsilon = 1 + \epsilon_2 - (1+\epsilon_1)^{-1},$$

then $K = MS^{-1}$.

V. EXAMPLES

In this section, we consider one academical example to discuss validity of conditions of theorems 1 and 2:

$$A_0 = \frac{1}{10}\begin{bmatrix} -6 & 5 & 4 \\ 7 & 5 & 2 \\ 1 & 5 & 3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}},$$
$$\overline{\Delta A} = \zeta\begin{bmatrix} 0.1 & 1 & 1 \\ 1 & 1 & 0.1 \\ 1 & 0.1 & 1 \end{bmatrix},\ \underline{\Delta A} = -\overline{\Delta A},$$
$$V = 0.01,\ v_t = V\sin(0.1t),$$

where $\zeta > 0$ is a parameter. The matrix A_0 has a negative element and it is not Schur stable.

For $\zeta = 0.02$ consider the problem of interval estimation using Theorem 1, then

$$\underline{L} = \overline{L} = [-0.6\ 0.7\ 0.1]^{\mathrm{T}}$$

is a solution of LMIs (10), (11) with

$$P = \mathrm{diag}[0.2824\ 1.3182\ 0.7895\ 0.338\ 1.946\ 1.1173].$$

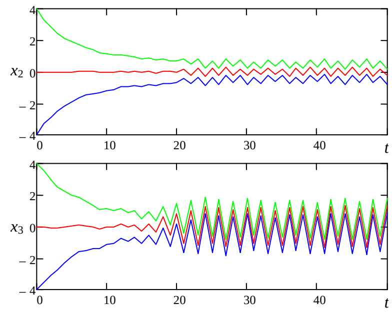

Figure 1. The results of simulations for the case of estimation

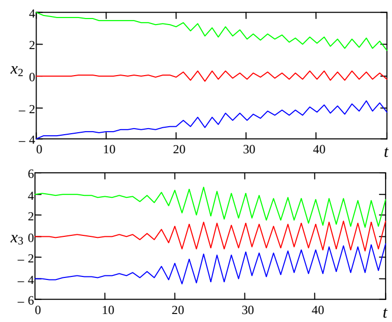

Figure 2. The results of simulations for the case of estimation and stabilization

The results of the system simulation for $u_t = -y_t$ are given in Fig. 1.

For $\zeta = 0.1$ consider the problem of output stabilization from Theorem 2, when the same $\underline{L}$, $\overline{L}$ are solutions of LMI (13). However, in this case the interval observer (4) with $u_t = -y_t$ is unstable (capabilities of the observer gains are not enough to compensate the uncertainty), then

$$\begin{aligned}\underline{K} &= [0.0002 \ -0.2552 \ -0.1508],\\ \overline{K} &= [0.0043 \ -0.2742 \ -0.1603]\end{aligned}$$

is a marginally feasible solution of LMI (14). The results of estimation and stabilization of the system with control (12) are given in Fig. 2.

VI. Conclusion

The problem studied in this paper is that of interval state estimation and robust stabilization of discrete-time LPV systems with unmeasurable vector of scheduling parameters. For solution of both problems different conditions of cooperativity and stability are expressed in terms of LMIs. The efficiency of the proposed observers is demonstrated on numerical simulations. Further investigations are necessary to reduce the conservatism of the proposed LMIs.

References

[1] J. Shamma, *Control of Linear Parameter Varying Systems with Applications*, ch. An overview of LPV systems, pp. 1–22. Springer, 2012.

[2] A. Marcos and J. Balas, "Development of linear-parameter-varying models for aircraft," *J. Guidance, Control, Dynamics*, vol. 27, no. 2, pp. 218–228, 2004.

[3] J. Shamma and J. Cloutier, "Gain-scheduled missile autopilot design using linear parameter-varying transformations," *J. Guidance, Control, Dynamics*, vol. 16, no. 2, pp. 256–261, 1993.

[4] W. Tan, *Applications of Linear Parameter-Varying Control Theory*. PhD thesis, Dept. of Mechanical Engineering, University of California at Berkeley, 1997.

[5] T. Meurer, K. Graichen, and E.-D. Gilles, eds., *Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems*, vol. 322 of *Lecture Notes in Control and Information Sciences*. Springer, 2005.

[6] T. Fossen and H. Nijmeijer, *New Directions in Nonlinear Observer Design*. Springer, 1999.

[7] G. Besançon, ed., *Nonlinear Observers and Applications*, vol. 363 of *Lecture Notes in Control and Information Sciences*. Springer, 2007.

[8] L. Jaulin, "Nonlinear bounded-error state estimation of continuous time systems," *Automatica*, vol. 38, no. 2, pp. 1079–1082, 2002.

[9] M. Kieffer and E. Walter, "Guaranteed nonlinear state estimator for cooperative systems," *Numerical Algorithms*, vol. 37, pp. 187–198, 2004.

[10] B. Olivier and J. Gouzé, "Closed loop observers bundle for uncertain biotechnological models," *Journal of Process Control*, vol. 14, no. 7, pp. 765–774, 2004.

[11] M. Moisan, O. Bernard, and J. Gouzé, "Near optimal interval observers bundle for uncertain bio-reactors," *Automatica*, vol. 45, no. 1, pp. 291–295, 2009.

[12] T. Raïssi, G. Videau, and A. Zolghadri, "Interval observers design for consistency checks of nonlinear continuous-time systems," *Automatica*, vol. 46, no. 3, pp. 518–527, 2010.

[13] T. Raïssi, D. Efimov, and A. Zolghadri, "Interval state estimation for a class of nonlinear systems," *IEEE Trans. Automatic Control*, vol. 57, no. 1, pp. 260–265, 2012.

[14] D. Efimov, L. Fridman, T. Raïssi, A. Zolghadri, and R. Seydou, "Interval estimation for lpv systems applying high order sliding mode techniques," *Automatica*, vol. 48, pp. 2365–2371, 2012.

[15] F. Mazenc and O. Bernard, "Interval observers for linear time-invariant systems with disturbances," *Automatica*, vol. 47, no. 1, pp. 140–147, 2011.

[16] C. Combastel, "Stable interval observers in c for linear systems with time-varying input bounds," *Automatic Control, IEEE Transactions on*, vol. PP, no. 99, pp. 1–6, 2013.

[17] D. Efimov, T. Raïssi, S. Chebotarev, and A. Zolghadri, "Interval state observer for nonlinear time varying systems," *Automatica*, vol. 49, no. 1, pp. 200–205, 2013.

[18] F. Mazenc, T. N. Dinh, and S. I. Niculescu, "Interval observers for discrete-time systems," in *Proc. IEEE CDC 2012*, 2012.

[19] D. Efimov, W. Perruquetti, T. Raïssi, and A. Zolghadri, "On interval observer design for discrete systems," in *Proc. ECC 2013,*, 2013.

[20] M. Krstić and H. Deng, *Stabilization of nonlinear uncertain systems*. Communications and Control Engineering, Springer, 1998.

[21] J. Mohammadpour and C. W. Scherer, eds., *Control of Linear Parameter Varying Systems with Applications*. Springer, 2012.

[22] D. Efimov, T. Raïssi, and A. Zolghadri, "Control of nonlinear and lpv systems: interval observer-based framework," *IEEE Trans. Automatic Control*, vol. 58, pp. ?–?, 2013.

[23] M. W. Hirsch and H. L. Smith, "Monotone maps: a review," *J. Difference Equ. Appl.*, vol. 11, no. 4-5, pp. 379–398, 2005.

[24] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. New York: Wiley, 2000.

[25] J. Löfberg, "Automatic robust convex programming," *Optimization methods and software*, vol. 27, no. 1, pp. 115–129, 2012.